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INTRODUCTION TO THE THIRD EDITION

This third edition of *Core Maths for Advanced Level* has been extensively revised to cover the pure mathematics for AS and A Levels in Mathematics based on the subject criteria specified by QCA. The material is dealt with in a progressive and logical way so that the knowledge and skills acquired for AS stage are used and developed in later topics for A2. Because this book is not limited to any specific scheme, it also covers material that some boards may not examine. Specific syllabuses need to be looked at in order to identify topics that are optional: a syllabus map is included.

As a starting point, the book assumes the minimum level of success on the national curriculum for access to A-level. Many of you will have reached a higher level, particularly in algebraic skills and so will find an overlap between some work in this book and what you already know. For you the early chapters provide useful revision but they also contain some work that you are unlikely to have covered. We suggest using the mixed exercises at the ends of these chapters to identify unfamiliar topics, so that you can, if you wish, restrict your study to these sections.

All too many students regard A-level mathematics as being intrinsically difficult. We strongly disagree with this opinion. Part of the reason for this myth may be that students, at an early stage in their course, tackle problems that are too sophisticated. The exercises in this book are designed to overcome this problem, all starting with straightforward questions. There are many A-level examination questions at regular intervals throughout the book. These extensive exercises are intended for use at a later date, to give practice in examination questions when confidence and sophistication have been developed. The summary sections also include a brief recap of the work in preceding chapters and a set of multiple choice questions, which are useful for self-testing even if they do not form part of the examination to be taken.

There are many computer programs that help with the understanding of mathematics. In particular, good graph drawing software is invaluable for investigating graphical aspects of functions. Graphics calculators are also invaluable and some of them are now available with facilities to link them to computers and printers. However, as their use is forbidden in certain examination papers, you should not get into the habit of relying on them too much.

Another very valuable aid, particularly for investigating sequences and analysing data, is a computer spreadsheet. In a few places we have indicated where such aids can be used effectively, but this should be regarded as a minimum indication of the possible use of technology.

We would like to thank Alison Gee for her thorough work in checking the book.

We are grateful to the following examination boards for giving their permission to reproduce questions from past examination papers.

London Examinations, a division of Edexcel Foundation (Edexcel)
Oxford, Cambridge and RSA Examinations (OCR)
Assessment and Qualifications Alliance (AQA)
Welsh Joint Education Committee (WJEC)

L. Bostock
S. Chandler
2000

CONTENTS

CHAPTER 1

Algebra 1

Skill in manipulating algebraic expressions is essential in any mathematics course beyond GCSE and needs to be almost as instinctive as the ability to manipulate simple numbers. This and the next two chapters present the facts and provide practice necessary for the development of these skills.

Multiplication of Algebraic Expressions

The multiplication sign is usually omitted, so that, for example,

$2q$ means $2 \times q$

and $x \times y$ can be simplified to xy

Remember also that if a string of numbers and letters are multiplied, the multiplication can be done in any order, for example

$$2p \times 3q = 2 \times p \times 3 \times q$$
$$= 6pq$$

Powers can be used to simplify expressions such as $x \times x$,

i.e. $x \times x = x^2$

and $x \times x^2 = x \times x \times x = x^3$

But remember that a power refers only to the number or letter it is written above, for example

$2x^2$ means that x is squared, but 2 is not.

EXAMPLE 1A

Simplify **a** $(4pq)^2 \times 5$ **b** $\dfrac{ax^2}{y} \div \dfrac{x}{ay^2}$

a $(4pq)^2 \times 5 = 4pq \times 4pq \times 5$
$= 80p^2q^2$

b $\dfrac{ax^2}{y} \div \dfrac{x}{ay^2} = \dfrac{ax^{\not 2}}{\not y} \times \dfrac{ay^{\not 2}}{\not x}$
$= a^2xy$

EXERCISE 1A

Simplify

1 $3 \times 5x$
2 $x \times 2x$
3 $(2x)^2$
4 $5p \times 2q$
5 $4x \times 2x$
6 $2pq \times 5pr$
7 $(3a)^2$
8 $7a \times 9b$
9 $8t \times 3st$
10 $2a^2 \times 4a$
11 $25x^2 \div 15x$
12 $12m^2 \div 6m$
13 $b^2 \times 4ab$
14 $25x^2y \div 5x$
15 $(7pq)^2 \times (2p)^2$
16 $\dfrac{22ab}{11b}$
17 $\dfrac{18ax^2}{3x}$
18 $\dfrac{36xy}{18y}$
19 $\dfrac{72ab^2}{40a^2b}$
20 $\dfrac{2}{5} \div \dfrac{1}{x}$
21 $\dfrac{x^2}{y} \div \dfrac{y}{x}$

5 Expanding $(1+\sqrt{2})^3$ gives

A $3+3\sqrt{2}$ D $3+\sqrt{6}$

B $7+5\sqrt{2}$ E $1+2\sqrt{2}$

C $1+3\sqrt{2}$

6 $x^3-7x^2+7x+15$ has a factor

A $x+1$ C $2x-1$

B $x+15$ D $x-3$

7 x^3+8 has a factor

A $x-2$ C $x-8$

B $x+2$ D x^2+2x+4

8 If $x^2+px+6=0$ has equal roots and $p>0$, p is

A $\sqrt{48}$ C $\sqrt{6}$ E $\sqrt{24}$

B 0 D 3

9 If $x^2+4x+p \equiv (x+q)^2+1$, the values of p and q are

A $p=5, q=2$ D $p=-1, q=5$

B $p=1, q=2$ E $p=0, q=-1$

C $p=2, q=5$

10 $\dfrac{p^{-1/2}\times p^{3/4}}{p^{-1/4}}$ simplifies to

A 1 C $p^{3/4}$ E $p^{1/2}$

B $p^{-1/2}$ D p

11 In the expansion of $(a-2b)^3$ the coefficient of b^2 is

A $-2a^2$ C $12a$ E -12

B $-8a$ D $-4a$

12 If $\log_x y = 2$ then

A $x=2y$ C $x^2=y$ E $y=\sqrt{x}$

B $x=y^2$ D $y=2x$

13 $2\ln x + \frac{1}{2}\ln y =$

A $\ln\dfrac{x^2}{2y}$ C $\ln(\sqrt{y}\times x^2)$

B $\ln x^2 + \ln\sqrt{y}$ D $\ln 2x + \ln\frac{1}{2}y$

14 $\log_{10}5 - 2\log_{10}2 + \frac{3}{2}\log_{10}16$ is equ to

A $\log_{10}80$ D $2\log_{10}12$

B 10 E $1+\log_{10}8$

C 0

15 When $(3-5x)^4$ is expanded

A the coefficient of x^4 is 1.

B the coefficient of x is -540.

C there are four terms after all simplification.

16 $y = \ln x - \ln 4$

A $y=\dfrac{x}{4}$ B $x=4e^y$ C $y = \ln$

17 $x^2-2x+2=$

A $(x-1)^2+1$ C $(x-1)^2$

B 0 when $x=1$

18 $\frac{1}{2}\log_4 16 - 1$

A is equal to zero.

B is equal to $\log_4 7$.

19 $\dfrac{2\sqrt{3}-2}{2\sqrt{3}+2}$

A can be expressed as a fraction with a rational denominator.

B is an irrational number.

C is equal to -1.

In questions 20 to 23, write T if the statement is true and F if the statement is fa

20 If $x-a$ is a factor of x^2+px+q, the equation $x^2+px+q=0$ has a root equal to a.

21 $3\log x + 1 = \log 10x^3$ is an equation.

22 In the expansion of $(1+x)^6$ th coefficient of x is 6.

23 As $\sqrt{x}=4$ gives $x^2=16$, th equation $\sqrt{x}=4$ has two sol

EXAMINATION QUESTIONS A

1 Determine the value of the rational number p for which

$$\frac{3^{1/4}\times 3\times 3^{1/6}}{\sqrt{3}} = 3^p$$ (OCR)

2 Show that the elimination of x from the simultaneous equations

$$x-2y=1$$
$$3xy-y^2=8$$

produces the equation $5y^2+3y-8=0$.

Solve this equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied. (Edexcel)

3 Express $(a^4)^{-1/2}$ as an algebraic fraction in simplified form. (OCR)

4 Solve the simultaneous equations

$$x+y=1,\quad x^2-xy+y^2=7.$$ (OCR)

5 Express $\dfrac{1}{(\sqrt{a})^{\frac{4}{3}}}$ in the form a^n, stating the value of n. (OCR)

6 Solve the simultaneous equations

$$2x+y=3,\quad 2x^2-xy=10.$$ (OCR)

7 $y=5x^3+24x^2+29x+2$.

a Use the factor theorem to find one factor of y.

b Hence write y in the form

$$(x+k)(ax^2+bx+c),$$

giving the value of each of the constants k, a, b, and c.

c Hence find the exact solutions to the equation $y=0$. (AQA)

8 It is given that $(x+2)$ is a factor of x^4-4x^2+2x+a. Find the value of the constant a. (OCR)

9 Express each of the following in the form $p+q\sqrt{7}$ where p and q are rational numbers.

a $(2+3\sqrt{7})(5-2\sqrt{7})$ b $\dfrac{(5+\sqrt{7})}{(3-\sqrt{7})}$ (AQA)

10 a Use an algebraic method to solve the simultaneous equations

$$y=x^2-3x+2 \text{ and } y=3x-7.$$

b Interpret your answer geometrically. (AQA)

11 Express $\dfrac{5(x-3)(x+1)}{(x-12)(x+3)} - \dfrac{3(x+1)}{x-12}$ as a single fraction in its simplest form. (Edexcel)

12 Use the laws of logarithms to express

$$3\ln 4 - \ln 24 + \tfrac{1}{2}\ln 2.25$$

as a single logarithm in its simplest form. Show all your working. (OCR)

13 a Write down the exact value of x given that $4^x = 8$.
b Use logarithms to find y, correct to 3 decimal places, when $5^y = 10$. (OC

14 a Expand $(1+x^2)(1+x^3)$, arranging your answers in ascending powers of x.
b Find, as a decimal number, the exact value of $(1+x^2)(1+x^3)$ for $x = 10^{-3}$. (Edexc

15 Write $\ln x^3 + \ln xy - \ln y^3$ as single term.
Hence obtain an expression for y in terms of x if $\ln x^3 + \ln xy - \ln y^3 = 0$ (AC

16 Express

$$x^2 - 8x - 3$$

in completed square form.
Hence, or otherwise, find the exact solutions of the equation $x^2 - 8x - 3 = 0$. (AC

17 Solve the equation $4x^3 + 12x^2 + 5x - 6 = 0$. (WJE

18 One root of the equation $x^3 + kx + 11 = 0$, where k is a constant, is 1. Find the value of k.
Hence find the other two roots of the equation, giving your answer in an exact form. (OC

19 Simplify $\dfrac{2+\sqrt{2}}{2-\sqrt{2}}$ expressing your answer in surd form. (WJE

20 Given that $p = e^x$ and $q = e^y$, express each of

a e^{x+y} **b** e^{2x-y}

in terms of p and q. Your answers must not involve either logarithms or powers of e. (OC

21 Given that $y = \log_b 45 + \log_b 25 - 2\log_b 75$, express y as a single logarithm in base b.
In the case when $b = 5$, state the value of y. (AC

22 Express $\log_2(x+2) - \log_2 x$ as a single logarithm.
Hence solve the equation $\log_2(x+2) - \log_2 x = 3$ (OC

23 Given that $y = 10^x$, show that

a $y^2 = 100^x$, **b** $\dfrac{y}{10} = 10^{x-1}$

c Using the results from **a** and **b** write the equation

$$100^x - 10\,001(10^{x-1}) + 100 = 0$$

as an equation in y.
d By first solving the equation in y, find the values of x which satisfy the given equation in x. (Edexc

24 It is given that

$$(x+a)(x^2+bx+2) \equiv x^3 - 2x^2 - x - 6,$$

where a and b are constants. Find the value of a and the value of b. (OC

25 The cubic polynomial $x^3 - 2x^2 - 2x + 4$ has a factor $(x-a)$, where a is an integer.
i Use the factor theorem to find the value of a.
ii Hence find exactly all three roots of the cubic equation $x^3 - 2x^2 - 2x + 4 = 0$. (OC

26 Explaining each step clearly, express

$$\log_2(8\sqrt{3}) - \tfrac{1}{3}\log_2 \tfrac{9}{16}$$

in the form $p + q\log_2 3$, where p and q are rational numbers to be found. (Edexc

27 $y = 2x^3 + 5x^2 - 8x - 15$.
a Show that when $x = -3$, $y = 0$.
b Hence factorise $2x^3 + 5x^2 - 8x - 15$.
c Find, to 2 decimal places, the two other values of x for which $y = 0$. (Edexcel)

28 Prove that $x^2 + px + q$ is a perfect square if, and only if, $p^2 = 4q$. (Edexcel)

29 Use the factor theorem to find one of the factors of the cubic

$$2x^3 - 9x^2 + 7x + 6.$$

Hence factorise the cubic into its linear factors. (WJEC)

30 Given that for all values of x,

$$3x^2 + 12x + 5 \equiv p(x+q)^2 + r,$$

a find the values of p, q and r.
b Hence, or otherwise, find the minimum value of $3x^2 + 12x + 5$.
c Solve the equation $3x^2 + 12x + 5 = 0$, giving your answers to one decimal place. (Edexcel)

31 a Given that a and b are positive numbers, show that

$$\ln(ab) = \ln a + \ln b.$$

b Express $\ln 6 - \ln 4 + \ln 8 - \ln 3$ as a single logarithm. (AQA)

32 Express $(2x+1)(x-2) - 3$ as a product of linear factors. (OCR)

33 Solve the simultaneous equations

$$x + y = 2, \quad x^2 + 2y^2 = 11$$ (OCR)

34 Given that k is a real constant such that $0 < k < 1$, show that the roots of the equation

$$kx^2 + 2x + (1-k) = 0$$

are **a** always real **b** always negative. (Edexcel)

35 The polynomial $x^3 + ax + b$ has $x - 1$ and $x - 3$ as two of its factors. Use this fact to write down two equations for a and b, and solve them. Hence find the third factor. (OCR)

36 Given that $(x-2)$ and $(x+2)$ are each factors of $x^3 + ax^2 + bx - 4$, find the values of a and b.
For these values of a and b, find the other linear factor of $x^3 + ax^2 + bx - 4$. (OCR)

37 Show that both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $x^4 + x^3 - x^2 - 3x - 6$.
Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor of this polynomial. (OCR)

38 a Assuming that $a = e^{\ln a}$ where $a > 0$, prove that

$$\ln(a^n) = n\ln a$$

b Find, correct to three decimal places, the value of y given that

$$2^{y+1} = 3 \times 5^y$$ (WJEC)

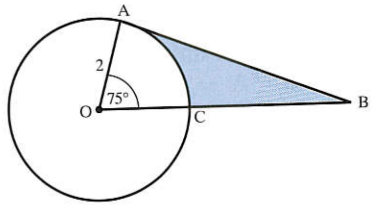
EXAMINATION QUESTIONS B

1 **a** **i** Find the gradient of the straight line $2x + 3y = -11$.

ii Find the equation of the line through $(9, -1)$ perpendicular to $2x + 3y = -11$.

b Calculate the coordinates of the point where these two lines meet. (OC

2



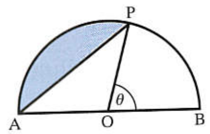
The diagram shows a circle of centre O and of radius 2 cm (not drawn to scale). The line AB is th tangent to the circle at A and the line OB cuts the circle at C. The angle AOB $= 75°$. Calculate th area of the shaded region correct to three significant figures. (AC

3 The points A, B and C have coordinates $(5, -3)$, $(7, 8)$ and $(-3, 4)$ respectively. The midpoint of BC is M.

a Write down the coordinates of M.

b Find the equation of the straight line which passes through the points A and M. (AC

4



The diagram shows a semicircle APB on AB as diameter. The midpoint of AB is O. The point P o the semicircle is such that the area of the sector POB is equal to twice the area of the shaded segment. Given that angle POB is θ radians, show that

$$3\theta = 2(\pi - \sin\theta).$$

(O

5 The vertices of the triangle ABC are $A(-3, 1)$, $B(10, -8)$ and $C(1, 4)$. Find an equation of the li passing through A and B, giving your answer in the form $px + qy + r = 0$, where p, q and r are integers. Show by calculation that CA and CB are perpendicular. (O

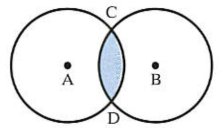
6 In triangle ABC, angle A $= 42°$, AB $= 6.0$ cm and BC $= 4.5$ cm. Calculate the two possible valu of angle C. (O

7 The points $A(-2, 4)$, $B(6, -2)$ and $C(5, 5)$ are the vertices of triangle ABC and D is the midpoi of AB.

a Find the equation of the line passing through A and B in the form $y = mx + c$, where the constants m and c are to be found.

b Show that CD is perpendicular to AB. (Ede

8 Two circles with centres A and B, each of radius 13 cm, lie in a plane with their centres 24 cm apart. The circles intersect at the points C and D.



Not to scale

a Determine the length of CD.

b Calculate the size of angle CAD in radians, giving your answer to four significant figures.

c Calculate the area of the shaded region common to both circles, giving your answer to the nearest $0.1\ \text{cm}^2$. (AQA)

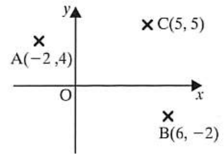
9 A, B, C are the points $(4, 3)$, $(2, 2)$ and $(5, -4)$ respectively.

a Show that the lines AB and BC are perpendicular.

b A point D is such that ABCD is a rectangle. Find the equation of the line AD and the equation of the line CD. Hence, or otherwise, find the coordinates of D.

c Find the area of rectangle ABCD. (WJEC)

10 The diagram shows the points $A(-2, 4)$, $B(6, -2)$ and $C(5, 5)$.



Not to scale

a Find the equation of the line passing through the points A and B giving your answer in the form $y = mx + c$ where the values of m and c are to be found.

b The point D is the midpoint of AB. Prove that CD is perpendicular to AB.

c Show that the line through C parallel to AB has equation $3x + 4y = 35$. (AQA)

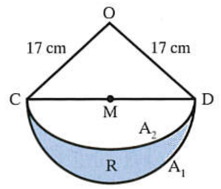
11 The straight line passing through the point $P(2, 1)$ and $Q(k, 11)$ has gradient $-\frac{5}{12}$.

a Find the equation of the line in terms of x and y only.

b Determine the value of k.

c Calculate the length of the line segment PQ. (Edexcel)

12



The figure shows the triangle OCD with OC $=$ OD $= 17$ cm and CD $= 30$ cm. The midpoint of CD is M. With centre M, a semicircular arc A_1 is drawn on CD as diameter. With centre O and radius 17 cm, a circular arc A_2 is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places,

a the area of the triangle OCD

b the angle COD in radians

c the area of the shaded region R. (Edexcel)

13 a Find an equation of the line l which passes through the points A$(1, 0)$ and B$(5, 6)$.

The line m with equation $2x + 3y = 15$ meets l at the point C.

b Determine the coordinates of C.

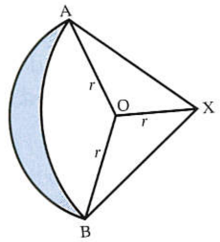
The point P lies on m and has x-coordinate -3.

c Show, by calculation, that PA = PB.

(Edexc

14 The left edge of the shaded crescent-shaped region, shown in the figure below, consists of an arc of a circle of radius r cm with centre O. The angle AOB $= \frac{2}{3}\pi$ radians.

The right edge of the shaded region is a circular arc with centre X, where OX $= r$ cm.



a Show that angle AXB $= \frac{1}{3}\pi$ radians.

b Show that AX $= r\sqrt{3}$ cm.

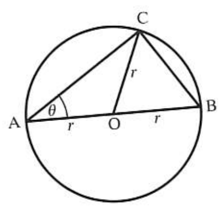
c Calculate, in terms of r, π and $\sqrt{3}$, the area of the shaded region.

(O

15 The straight line p passes through the point $(10, 1)$ and is perpendicular to the line r with equation $2x + y = 1$. Find the equation of p. Find also the coordinates of the point of intersect of p and r, and deduce the perpendicular distance from the point $(10, 1)$ to the line r.

(O

16 The figure shows a circle with centre O and radius r. Points A, B and C lie on the circle such that AB is a diameter. Angle BAC $= \theta$ radians.

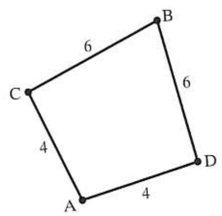


a Write down the size of angle AOC in terms of θ.

b Use the cosine rule in triangle AOC to express AC^2 in terms of r and θ.

c By considering right-angled triangle ABC, write down the length of AC in terms of r and θ. Deduce that $\cos 2\theta = 2\cos^2\theta - 1$.

17 The points A and B have coordinates $(3, -1)$ and $(6, 3)$ respectively. The points C and D are each distant 4 units from A and 6 units from B, as shown in the diagram.

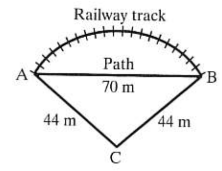


a Calculate the length AB.

b Calculate the cosine of angle CAB.

c Show that the length of CD is $3\sqrt{7}$.

18



There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the figure.

a Show that the size, to 2 decimal places, of ∠ACB is 1.84 radians.

Calculate

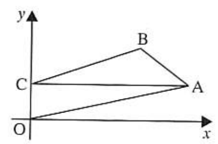
b the length of the railway track

c the shortest distance from C to the path

d the area of the region bounded by the railway track and the path.

(Edexcel)

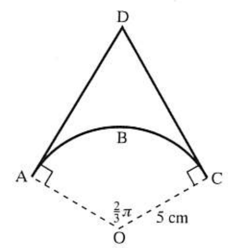
19



The diagram, not drawn to scale, shows a trapezium OABC with OA parallel to CB. Given that B is the point $(4, 3)$, C is the point $(0, 2)$ and the diagonal CA is parallel to the x-axis, **calculate** the coordinates of A.

(OCR)

20



In the diagram, ABC is an arc of a circle with centre O and radius 5 cm. The lines AD and CD are tangents to the circle at A and C respectively. Angle AOC $= \frac{2}{3}\pi$ radians. Calculate the area of the region enclosed by AD, DC and the arc ABC, giving your answer correct to 2 significant figures.

(OCR)

21 The line L passes through the points A$(1, 3)$ and B$(-19, -19)$.

a Calculate the distance between A and B.

b Find an equation of L in the form $ax + by + c = 0$, where a, b and c are integers.

(Edexcel)

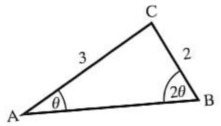
22 a Find an equation of the straight line passing through the points with coordinates $(-1, 5)$ and $(4, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line crosses the x-axis at the point A and the y-axis at the point B, and O is the origin.

b Find the area of △OAB.

(Edexcel)

23

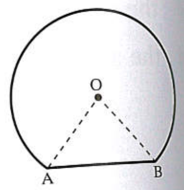


In the triangle ABC, AC = 3 cm, BC = 2 cm, $\angle BAC = \theta$ and $\angle ABC = 2\theta$.
Calculate the value of θ correct to the nearest tenth of a degree.

Hence find the size of the angle ACB and, without further calculation, explain why the length of AB is greater than 2 cm. (AQ

24 Find the equation of the straight line that passes through the points $(3, -1)$ and $(-2, 2)$, giving your answer in the form $ax + by + c = 0$. Hence find the coordinates of the point of intersection of the line and the x-axis. (OC

25 The diagram shows the cross-section of a tunnel.
The cross-section has the shape of a major segment of a circle, and the point O is the centre of the circle.
The radius of the circle is 4 m, and the size of angle AOB is 1.5 radians.
Calculate the perimeter of the cross-section. (OC



26 The points P, Q and R have coordinates $(2, 4)$, $(7, -2)$ and $(6, 2)$ respectively. Find the equation of the straight line l which is perpendicular to the line PQ and which passes through the midpoint of PR. (AQ

27 The points A and B have coordinates $(8, 7)$ and $(-2, 2)$ respectively. A straight line l passes through A and B and meets the coordinate axes at the points C and D.

a Find, in the form $y = mx + c$, the equation of l.

b Find the length CD, giving your answer in the form $p\sqrt{q}$, where p and q are integers and q is prime. (Edex

28 The line l has equation $2x - y - 1 = 0$.
The line m passes through the point A $(0, 4)$ and is perpendicular to the line l.

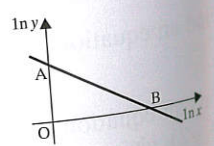
a Find an equation of m and show that the lines l and m intersect at the point P $(2, 3)$.

The line n passes through the point B $(3, 0)$ and is parallel to the line m.

b Find an equation of n and hence find the coordinates of the point Q where the lines l and n intersect. (Ed

c Prove that AP = BQ = PQ.

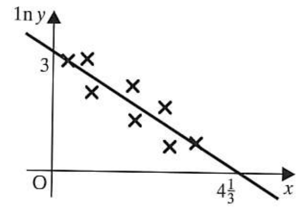
29 The figure shows a straight line graph of $\ln y$ against $\ln x$.
The line crosses the axes at A $(0, 3)$ and B $(3.5, 0)$.



a Find an equation relating $\ln y$ and $\ln x$.

b Hence, or otherwise, express y in the form px^q, giving the values of the constants p and q to 3 significant figures. (Ed

30 The figure shows a graph of $\ln y$ against x for two sets of observations x and y. The line of best fit is shown and it crosses the x-axis at $x = 4\frac{1}{3}$ and the $(\ln y)$-axis at $\ln y = 3$



a Find an equation for the line of best fit.

b Express y as a function of x in the form $y = ab^x$ where a and b are constants. Write down the values of a and b to 1 significant figure. (Edexcel)

31 A zoo keeps an official record of the mass, x kg, and the average daily food intake, y kg, of each adult animal. Four selected pairs of values of x and y are given in the table below.

Animal	Cheetah	Deer	Rhinoceros	Hippopotamus
x	40	170	1500	3000
y	1.8	5.0	33.2	50.0

Show, by drawing a suitable linear graph on a sheet of 2 mm graph paper, that these values are approximately consistent with a relationship between x and y of the form

$y = ax^m$,

where a and m are constants.

Use your linear graph to find an estimate of the value of m.

The zoo has a bear with mass 500 kg. Assuming that this animal's food intake and mass conform to the relationship mentioned above, indicate a point on your linear graph corresponding to the bear and estimate the bear's average daily food intake. (AQA)

32 The variables x and y satisfy a relationship of the form $y = ax^b$, where a and b are constants.
Measurements of y for given values of x gave the following results.

x	2	3	4	5	6
y	6.32	7.24	7.98	8.60	9.12

a Plot $\ln y$ against $\ln x$ and draw the line of best fit to the plotted points.

b Use your line to estimate

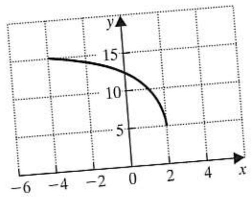
i the value of x when $y = 7.50$ giving your answer correct to two significant figures.

ii the values of a and b, giving your answers to an appropriate degree of accuracy. (AQA)

Examination Questions C

1 Use calculus to find the values of x for which $y = 3x^4 + 8x^3 + 6x^2 - 1$ has stationary points. (AQA)

2



The entire graph of a function $y = \mathrm{f}(x)$ is illustrated above.

a Write down the domain of the function $\mathrm{f}(x)$.

b Sketch the graph of the inverse function $y = \mathrm{f}^{-1}(x)$ marking appropriate values on the axes.

c Write down the range of $\mathrm{f}^{-1}(x)$. (AQA)

3 Find the equation of the tangent to the curve $y = x^3 + 2x^2 + 3x + 6$ at the point where $x = -1$. (AQA)

4 Functions f and g are defined by

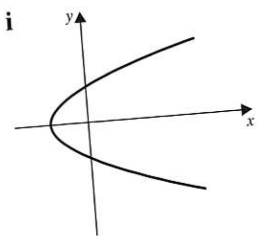
$$\mathrm{f}: x \mapsto \frac{3}{x+3}, \quad x \in \mathbb{R}, \quad x \geqslant 0,$$

$$\mathrm{g}: x \mapsto x + 1, \quad x \in \mathbb{R}, \quad x \geqslant 0,$$

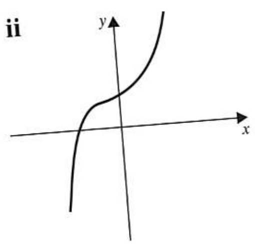
Show that $\quad \mathrm{gf}: x \mapsto \dfrac{x+6}{x+3}, \quad x \in \mathbb{R}, \quad x \geqslant 0$

Express fg in similar form. Find $(\mathrm{gf})^{-1}(x)$. (OCR)

5 **i**



ii



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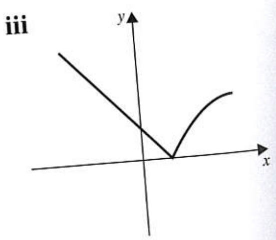
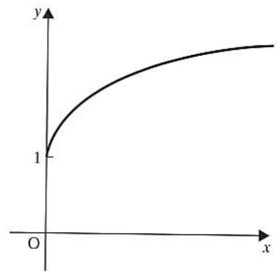


a For each of the graphs **i**, **ii** and **iii**, state whether or not it represents a function.

b One of the graphs above is such that the function f represented by the graph has an inverse, f^{-1}.
Assuming equal scales on the axes of the graphs drawn, sketch the graph of f^{-1}. (AQA)

6 Find, in degrees to 1 decimal place, the values of x which lie in the interval $-180° \leqslant x \leqslant 180°$ and satisfy the equation $\sin 2x = -0.57$. (Edexcel)

7 The diagram shows the graph of the function f defined for $x \geqslant 0$ by $\mathrm{f}: x \mapsto 1 + \sqrt{x}$.



Copy the sketch, and show on the same diagram the graph of f^{-1}, making clear the relation between the two graphs.

Give an expression in terms of x for $\mathrm{f}^{-1}(x)$, and state the domain of f^{-1}.
There is one value of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. By considering your diagram, explain why this value of x satisfies the equation $1 + \sqrt{x} = x$.

By treating the equation $1 + \sqrt{x} = x$ as a quadratic equation for $\sqrt{x}$, or otherwise, show that the value of x satisfying $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is $\frac{1}{2}(3 + \sqrt{5})$. (OCR)

8 Given that $\cos^2 x = \frac{2}{9}$, where $\pi \leqslant x \leqslant 2\pi$, find the exact value, or values, of

a $\sin x$ **b** $\tan x$ (Edexcel)

9 Let $\mathrm{f}(A) = \dfrac{\cos A}{1 + \sin A} + \dfrac{1 + \sin A}{\cos A}$.

a Prove that $\mathrm{f}(A) = 2 \sec A$.

b Solve the equation $\mathrm{f}(A) = 4$,
giving your answers for A, in degrees, in the interval $0° < A < 360°$. (OCR)

10 *No credit will be given for numerical answers without supporting working.*

Solve the equation

$$4 \cot^2 \theta + 12 \operatorname{cosec} \theta + 1 = 0$$

giving all answers of θ to the nearest degree in the interval $0° \leqslant \theta \leqslant 360°$. (AQA)

11 *No credit will be given for a numerical approximation or for a numerical answer without supporting working.*

a Find all solutions of the equation

$$\tan 2x = 2$$

in the range $0 \leqslant x \leqslant 360°$. Give your answers correct to one decimal place.

b Find all solutions of the equation

$$2 \sin^2 x + 3 \cos x = 0$$

in the interval $0 \leqslant x \leqslant 360°$. (WJEC)

12 A piece of wire of total length 12 m is cut into two pieces. One piece of wire is bent into a rectangle of sides x m and $3x$ m and the other is bent to form the boundary of a square. Show that the total area enclosed by the rectangle and the square is given by

$$A = 7x^2 - 12x + 9 \text{ m}^2.$$

Find the value of x if the total area enclosed is a minimum. Verify that your stationary value is a minimum. (WJEC)

13 The functions f and g are defined by

$$f(x) = 2\sqrt{x-5}, \quad x \in \mathbb{R}, x \geqslant 5,$$

$$g(x) = x^2 + 3, \quad x \in \mathbb{R}.$$

a Find the range of f and the range of g.

b Explain *briefly* why the function fg does not exist.

c Find
i an expression for $gf(x)$
ii the domain and the range of the function gf.

d Explain briefly why the inverse function of g does not exist.

e Find an expression for $f^{-1}(x)$, giving the domain and range of f^{-1}. (WJEC)

14 A curve has equation $y = x^3 + 3x^2 - 9x + 4$.

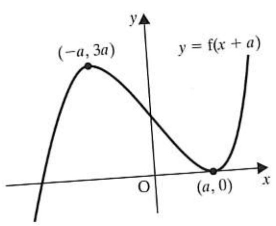
a Find $\dfrac{dy}{dx}$.

b Find the coordinates of the two stationary points on the curve.

c Use calculus to determine the nature of each of the stationary points and show these on a sketch of the curve.

d Deduce the range of k for which $x^3 + 3x^2 - 9x + 4 = k$ has three real roots. (WJEC)

15



The diagram shows the curve $y = f(x+a)$, where a is a positive constant. The maximum and minimum points on the curve are $(-a, 3a)$ and $(a, 0)$ respectively. Sketch the following curves, on separate diagrams, in each case stating the coordinates of the maximum and minimum points:

i $y = f(x)$, ii $y = -2f(x+a)$. (OCR)

16 A large tank in the shape of a cuboid is to be made from $54\,\text{m}^2$ of sheet metal. The tank has a horizontal rectangular base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a Show that the volume, $V\,\text{m}^3$, of the tank is given by

$$V = 18x - \tfrac{2}{3}x^3.$$

b Given that x can vary, use differentiation to find the maximum value of V.

c Justify that the value of V you have found is a maximum. (Edexcel)

17 The functions f and g are defined as follows:

$$f(x) = \frac{1}{x}\ (x \neq 0); \quad g(x) = \frac{1}{x+1}\ (x \neq -1)$$

a Find an expression, in terms of x, for $g^{-1}(x)$.

b Describe geometrically the transformation which maps
i the graph of $y = f(x)$ into the graph of $y = g(x)$
ii the graph of $y = f(x)$ into the graph of $y = g^{-1}(x)$.

c Find an expression, in terms of x, for the composite function $gf(x)$. Simplify your answer.

d Given that there is a function h such that $gh(x) = f(x)$, find $h(x)$. (OCR)

18 The specification for a new rectangular car park states that the length x m is to be 5 m more than the breadth. The perimeter of the car park is to be greater than 32 m.

a Form a linear inequality in x.

The area of the car park is to be less than $104\,\text{m}^2$.

b Form a quadratic inequality in x.

c By solving your inequalities, determine the set of possible values of x. (Edexcel)

19 A curve has equation $y = \dfrac{1}{x} - \dfrac{1}{x^2}$. Use differentiation to find the coordinates of the stationary point and determine, showing your working, whether the stationary point is a maximum point or a minimum point. Deduce, or obtain otherwise, the coordinates of the stationary points of each of the following curves:

i $y = \dfrac{1}{x} - \dfrac{1}{x^2} + 5$, ii $y = \dfrac{2}{x-1} - \dfrac{2}{(x-1)^2}$ (OCR)

20 Solve the equation $4\tan^2 x + 12\sec x + 1 = 0$, giving all solutions in degrees, to the nearest degree, in the interval $-180° < x < 180°$. (AQA)

21 Find, in degrees, the values of θ in the interval $[0, 360°]$ for which

$$4\sin^2\theta - 2\sin\theta = 4\cos^2\theta - 1.$$

In your answers, distinguish clearly between those that are exact and those which are given to a degree of accuracy of your choice, which you should state. (Edexcel)

22 A student models the evening lighting-up time by the equation

$$L = 6.125 - 2.25\cos\left(\frac{\pi t}{6}\right)$$

where the time, L hours pm, is always in GMT (Greenwich Mean Time) and t is in months, starting in mid-December. The model assumes that all months are equally long.

a Calculate the value of L for mid-January and for mid-May.

b Find, by solving an appropriate equation, the two months in the year when the lighting-up time will be 5 pm (GMT)

c Write down an equation for L if t were to be in months starting in mid-March. (AQA)

23 i Draw sketch graphs of $y = \sin x$ and $y = -\cos x$ for $0° \leqslant x \leqslant 360°$.

ii Mo needs to draw the graph of $y = 2\sin^2 x$, also for $0° \leqslant x \leqslant 360°$, and makes two attempts, P and Q shown in the figure below.

Realising that both cannot be right, Mo tests each of P and Q by finding the value of y when $x = 270°$.

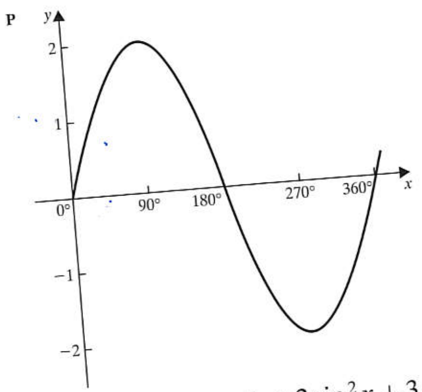
Carry out the test for each of P and Q. In each case state which one of the statements A, B, C below is a correct conclusion from the test.

A The curve is correctly drawn.

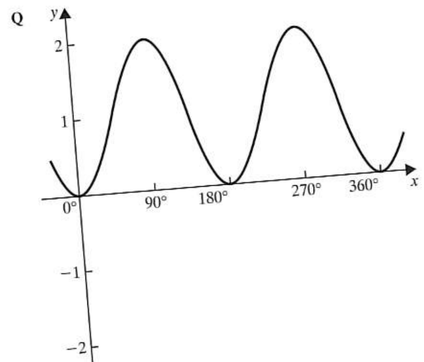
B The curve is incorrectly drawn.

C There is no evidence that the curve is incorrectly drawn.

P



Q



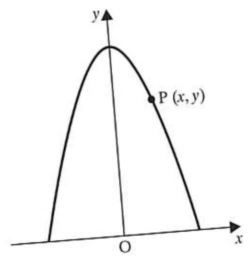
iii a Solve the equation $2\sin^2 x + 3\cos x = 0$ for $0° \leqslant x \leqslant 360°$.

b Given that one of the curves, P and Q, in part **ii** is in fact correct, illustrate your answers to part **iii a** by drawing two suitable curves. (OCR)

24 a Given that $\tan 75° = 2 + \sqrt{3}$, find in the form $m + n\sqrt{3}$, where m and n are integers, the value of **i** $\tan 15°$ **ii** $\tan 105°$

b Find, in radians to two decimal places, the values of x in the interval $0 \leqslant x \leqslant 2\pi$, for which $3\sin^2 x + \sin x - 2 = 0$. (Edexcel)

25



The figure shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y \geqslant 0$.

The point P(x, y) lies on the curve and O is the origin.

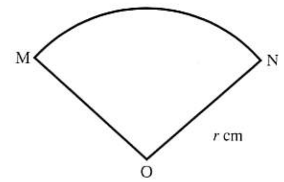
a Show that $\text{OP}^2 = \frac{1}{4}x^4 - 4x^2 + 25$.

Taking $f(x) \equiv \frac{1}{4}x^4 - 4x^2 + 25$,

b find the values of x for which $f'(x) = 0$.

c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. (Edexcel)

26 The figure shows a minor sector OMN of a circle centre O and radius r cm. The perimeter of the sector is 100 cm and the area of the sector is A cm^2.



a Show that $A = 50r - r^2$.

Given that r varies, find

b the value of r for which A is a maximum and show that A is a maximum

c the value of $\angle$MON for this maximum area

d the maximum area of the sector OMN. (Edexcel)

27 Given that $y = x^3 - x + 6$,

a find $\dfrac{dy}{dx}$.

On the curve representing y, P is the point where $x = -1$.

b Calculate the y-coordinate of the point P.

c Calculate the value of $\dfrac{dy}{dx}$ at P.

d Find the equation of the tangent at P.

The tangent at the point Q is parallel to the tangent at P.

e Find the coordinates of Q.

f Find the equation of the normal to the curve at Q. (OCR)

28 Find the set of values of x for which $2(x^2 - 5) < x^2 + 6$ (AQA)

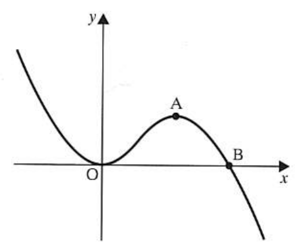
29 The curve with equation $y = 2 + k\sin x$ passes through the point with coordinates $(\frac{1}{2}\pi, -2)$.

Find **a** the value of k **b** the greatest value of y,

c the values of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $y = 2 + \sqrt{2}$. (Edexcel)

30 The diagram shows the graph of $y = x^2(3 - x)$.

The coordinates of the points A and B on the graph are $(2, 4)$ and $(3, 0)$ respectively.



a Write down the solution set of the inequality $x^2(3 - x) \leqslant 0$.

b The equation $3x^2 - x^3 = k$ has three real solutions for x. Write down the set of possible values for k.

c Functions f and g are defined as follows:

$$f : x \to x^2(3 - x), \quad 0 \leqslant x \leqslant 2,$$
$$g : x \to x^2(3 - x), \quad 0 \leqslant x \leqslant 3.$$

Explain why f has an inverse while g does not.

d State the domain and range of f^{-1}, and sketch the graph of f^{-1}. (OCR)

31 Find the set of values of x for which (Edexcel)

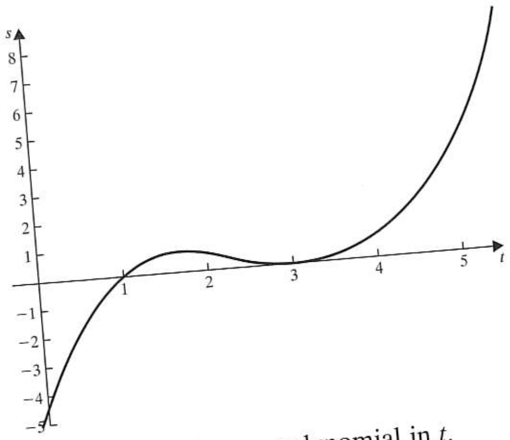
$$x^2 - x - 12 > 0.$$

32 Find, correct to the nearest degree, all the values of θ between $0°$ and $360°$ satisfying the equation (WJEC)

$$8\cos^2\theta + 2\sin\theta = 7.$$

33 **a** Write down an example of a polynomial in x of order 4.

b In an experiment, Ama measures the value of s at different times, t. Her results are shown as the curve on the graph below



Ama believes that it is possible to model s as a polynomial in t.

i Explain why it is reasonable to think that the order of such a polynomial might be 3.

Ama proposes a model of the form

$$s = a(t-p)(t-q)^2$$

ii Write down the points where the curve meets the coordinate axes and use them to find values for p, q and a.

iii Compare the values obtained from the model with those on the graph when $t = 2$, 4 and 5 and comment on the quality of the model.

Ama proposes a refinement to the model making it into

$$s = a(t-p)(t-q)^2(1-ht)$$

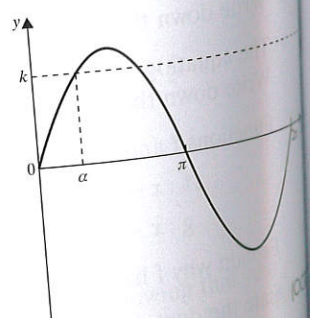
where a, p and q have the same values as before and h is a small positive constant. Ama chooses the value of h so that the model and the graph are in agreement when $t = 5$.

iv Find the value of h. (OC[R])

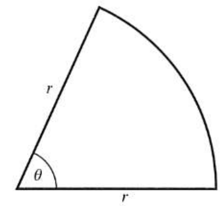
34 The diagram shows part of the graph of $y = \sin x$, where x is measured in radians, and the values of α on the x-axis and k on the y-axis are such that $\sin\alpha = k$. Write down, in terms of α,

a a value of x between $\frac{1}{2}\pi$ and π such that $\sin x = k$

b two values of x between 3π and 4π such that $\sin x = -k$. (O[CR])



35 A piece of wire of length 4 metres is bent into the shape of a sector of a circle of radius r metres and angle θ radian.



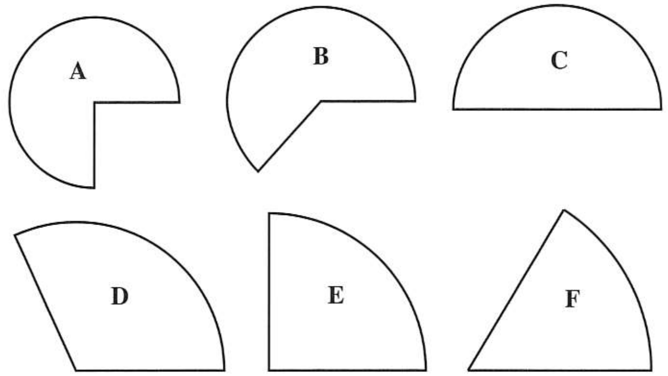
a State, in terms of θ and r.

i the length of the arc **ii** the area A of the sector.

b Hence show that $A = 2r - r^2$.

c Find the value of r which will make the area a maximum. Deduce the corresponding value of θ.

d The figures labelled A–F below show, all to the same scale, six possible sectors which can be made from the piece of wire. Which of them has the largest area?



(OCR)

36 The function f with domain $\{x : x \geqslant 0\}$ is defined by $f(x) = \dfrac{8}{x+2}$.

a Sketch the graph of f and state the range of f.

b Find $f^{-1}(x)$, where f^{-1} denotes the inverse of f.

c Calculate the value of x for which $f(x) = f^{-1}(x)$. (AQA)

37 Use differentiation to find the coordinates of the stationary points on the curve

$$y = x + \frac{4}{x},$$

and determine whether each stationary point is a maximum point or a minimum point. Find the set of values of x for which y increases as x increases. (OCR)

38 **a** Find the values of $\cos x$ for which

$$6\sin^2 x = 5 + \cos x.$$

b Find all the values of x in the interval $180° < x < 540°$ for which

$$6\sin^2 x = 5 + \cos x.$$

(Edexcel)

39 Given that $y = x^3 - 4x^2 + 5x - 2$, find $\dfrac{dy}{dx}$.

P is the point on the curve where $x = 3$.

a Calculate the y-coordinate of P.

b Calculate the gradient at P.

c Find the equation of the tangent at P.

d Find the equation of the normal at P.

Find the values of x for which the curve has a gradient of 5. (OCR)

40 A landscape gardener is given the following instructions about laying a rectangular lawn. The length x m is to be 2 m longer than the width. The width must be greater than 6.4 m and the area is to be less than 63 m^2.
By forming an inequality in x, find the set of possible values of x. (Edexcel)

41 Solve the equation

$$9\cos^2 x - 6\cos x - 0.21 = 0, \quad 0° \leqslant x \leqslant 360°,$$

giving each answer in degrees to 1 decimal place. (Edexcel)

42 Express $2x^2 + 5x + 4$ in the form $a(x+b)^2 + c$, stating the numerical values of a, b and c.
Hence, or otherwise, write down the coordinates of the minimum point on the graph of $y = 2x^2 + 5x + 4$. (OCR)

43 The function f is given by

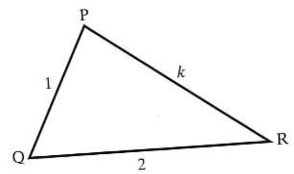
$$\text{f} : x \mapsto x^2 - 8x, \quad x \in \mathbb{R}, x \leqslant 4.$$

a Determine the range of f.

b Find the value of x for which $\text{f}(x) = 20$.

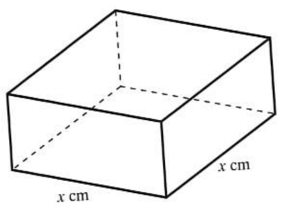
c Find $\text{f}^{-1}(x)$ in terms of x. (Edexcel)

44 The diagram shows triangle PQR, in which PQ = 1 unit, QR = 2 units and RP = k units.
Express $\cos R$ in terms of k.



Given that $\cos R < \frac{7}{8}$, show that $2k^2 - 7k + 6 < 0$. Find the set of values of k satisfying this inequality. (OCR)

45 The diagram shows a rectangular cake-box, with no top, which is made from thin card. The volume of the box is 500 cm^3. The base of the box is a square with sides of length x cm.



a Show that the area, A cm^2, of card used to make such an open box is given by

$$A = x^2 + \frac{2000}{x}.$$

b Given that x varies, find the value of x for which $\frac{dA}{dx} = 0$.

c Find the height of the box when x has this value.

d Show that when x has this value, the area of the card used is least. (Edexcel)

46 The function f is defined by

$$\text{f} : x \rightarrow \frac{3x+1}{x-2}, \quad x \in \mathbb{R}, x \neq 2$$

Find, in a similar form, the functions

a ff **b** f^{-1} (Edexcel)

47 Given that $\text{f}(x) = x^4$ and $\text{g}(x) = x + 2$, simplify
$\text{fg}(x) - \text{gf}(x)$ (AQA)

48 Given that $\text{f}(x) \equiv x^3 + 2x^2 - 5x - 6$, find

a $\text{f}(2)$

b the complete set of values of x for which $\text{f}(x) < 0$. (Edexcel)

49 Sketch the graph of the curve with equation $y = x(1-x)$. Determine the greatest and least values of y when $-1 \leqslant x \leqslant 1$. (OCR)

Examination Questions D

1 The equation of a curve is $y = 2x^2 - \ln x$, where $x > 0$. Find by differentiation the x-coordinate of the stationary point on the curve, and determine whether this point is a maximum point or a minimum point. (OCR)

2 a Sketch the graph of $y = e^x$ for all real values of x.

b On the same axes sketch the graph of $y = e^{-x}$. Describe a simple transformation which maps the graph of $y = e^x$ onto the graph of $y = e^{-x}$. (AQA)

3 a Sketch the graph of $y = e^x$.

b Given that $f(x) = e^x - x - 1$, show that f is an increasing function for $x > 0$. (WJEC)

4 The fourth term of an arithmetic series is 20 and the ninth term is 40. Find

a the common difference b the first term c the sum of the first 20 terms. (WJEC)

5 It is given that $f(x) \equiv (x - \alpha)(x - \beta), x \in \mathbb{R}$, where α and β are positive constants. Sketch, on separate diagrams, the curves with the following equations, giving in each case the coordinates of the points at which the curve meets the x-axis. (OCR)

i $y = f(x)$, ii $y = |f(x)|$, iii $y = f(x + 2\alpha)$

6 An arithmetic series has first term 82 and common difference -10.

i Show that the sum of the first n terms is

$$n(87 - 5n).$$

ii Find the value of n for which this sum is equal to 370. (AQA)

7 Find $\displaystyle\int_1^4 \left(x + \frac{6}{\sqrt{x}}\right)^2 dx$

No credit will be given for a numerical approximation or for a numerical answer without supporting working. (AQA)

8 The function f is defined by $f : x \mapsto 5 + 8x^2 - 36\ln x$ and has domain $x \geqslant 1$.

a Using calculus, determine the stationary value of f, giving your answer to 2 significant figures.

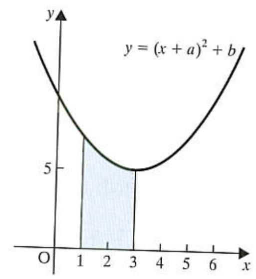
b Find the range of the function f″. (Edexcel)

9 Given that a is a positive constant, sketch the graph of $y = |3x - a|$, indicating clearly in terms of a where the graph crosses or touches the coordinate axes.
Solve the inequality $|3x - a| < x$. (AQA)

10 The graph of $y = (x + a)^2 + b$ is sketched opposite.

a Write down the values of a and b.

b Use algebraic integration to find the area of the shaded region shown.



(AQA)

11 The functions f and g are defined over the set of real numbers by

$$f : x \mapsto 3x - 5,$$
$$g : x \mapsto e^{-2x}.$$

a State the range of g.

b Sketch the graphs of the inverse functions f^{-1} and g^{-1} and write on your sketches the coordinates of any points at which a graph meets the coordinates axes.

c State, giving a reason, the number of roots of the equation $f^{-1}(x) = g^{-1}(x)$.

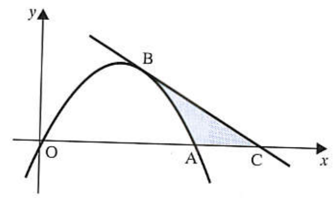
d Evaluate $fg(-\frac{1}{3})$, giving your answer to 2 decimal places. (Edexcel)

12 An arithmetic progression has 241 terms and a common difference of 0.1. Given that the sum of all the terms is 964, find the first term. (OCR)

13 The diagram shows the curve $y = 3x - x^2$. The curve meets the x-axis at the origin O and at the point A. The tangent to the curve at the point B(2, 2) intersects the x-axis at C.

a Find the equation of the tangent to the curve at B.

b Find the shaded area.



(WJEC)

14 Sketch the graph of $y = |x - 2a|$, where a is a positive constant. (You should indicate the coordinates of the points where the graph meets the axes.)
Find, in terms of a, the two values of x satisfying $|x - 2a| = \frac{1}{2}a$. (OCR)

15 The functions f and g are given by

$$f : x \mapsto 3x - 1, \ x \in \mathbb{R},$$
$$g : x \mapsto e^{\frac{x}{2}}, \ x \in \mathbb{R}.$$

a Find the value of fg(4), giving your answer to 2 decimal places.

b Express the inverse function f^{-1} in the form $f^{-1} : x \mapsto \ldots$

c Using the same axes, sketch the graphs of the functions f and gf. Write on your sketch the value of each function at $x = 0$.

d Find the values of x for which $f^{-1}(x) = \dfrac{5}{f(x)}$. (Edexcel)

16 a Given that the first and second terms of an arithmetic progression are 12 and 6 respectively, find the sum of the first hundred terms.

b Given that the first and second terms of a geometric progression are 12 and 6 respectively, show that the sum of the first ten terms is $\frac{3069}{128}$. (OCR)

17 An equation of a curve C is $y = \ln 3 + \ln x$.

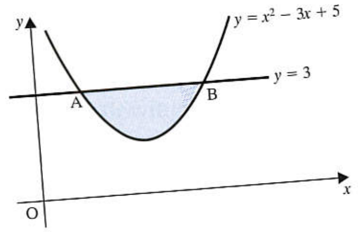
a Find the coordinates of the point where C crosses the x-axis.

b Sketch, in a single diagram, both C and the curve with equation $y = \ln x$.

c Express $\ln 3 + \ln x$ as a single logarithm, and hence show that the inverse function of the function $(\ln 3 + \ln x)$ is $\frac{1}{3}e^x$

d Sketch, in a single diagram, the graphs of $y = e^x$ and $y = \frac{1}{3}e^x$, and describe briefly the relationship between these graphs and the graphs you sketched in answer to part (b). (Edexcel)

18



The graph shows sketches of the line $y = 3$ and the curve $y = x^2 - 3x + 5$ (not drawn to scale); they intersect at the points A and B. The shaded region is bounded by the arc AB and the chord AB.

a Find the coordinates of A and B.

b Find the area of the shaded region.

c Show that the equation of the tangent to the curve at A is

$$y + x - 4 = 0$$

and find the equation of the tangent to the curve at B.

d The tangents to the curve at A and B meet at the point C. Show that the coordinates of C are $\left(\frac{3}{2}, \frac{5}{2}\right)$. (AQA)

19 An arithmetic series has common difference 1.6 and first term 6. The sum of the first n terms is denoted by S_n.

a Show that S_n may be expressed in the form $pn^2 + qn$ for some constants p and q. State the value of p and the value of q.

b Find the value of the positive integer n for which $S_n = n^2$. (OCR)

20 The function f is defined by $f : x \mapsto e^x + k$, $x \in \mathbb{R}$ and k is a positive constant.

a State the range of f.

b Find $f(\ln k)$, simplifying your answer.

c Find f^{-1}, the inverse function of f, in the form $f^{-1} : x \mapsto \ldots$, stating its domain.

d On the same axes, sketch the curves with equations $y = f(x)$, and $y = f^{-1}(x)$, giving the coordinates of all points where the graphs cut the axes. (Edexcel)

21 A young person decides to save £50 at the start of each month to supplement her pension when she retires. Interest is calculated at the end of each month and is added to her account. The total in her account after n months can be modelled by the expression

$$\sum_{i=1}^{n} 50 \times 1.004^i.$$

a Find the total amount in her account after 3 months. Give your answer to the nearest 10p.

b Calculate the total amount in her account if she continues this method of saving without a break for 35 years. Give your answer to the nearest £100.

c Find the annual rate of interest assumed in this model. Give your answer to 1 decimal place. (AQA)

22 A curve has equation $y = e^x - kx$ where k is a constant ($k > 0$).

a The curve has a single stationary point at M. Calculate the x-coordinate of M and hence show that the y-coordinate of M can be written as $k(1 - \ln k)$.

b Use the result from part (a) to determine the exact value of k for which the curve touches the x-axis.

c In the case when $k = 2$,

i find the value of $\frac{d^2y}{dx^2}$ at M and hence determine the nature of M. Deduce that the curve lies entirely above the x-axis.

ii calculate the area of the finite region bounded by the curve, the coordinate axes and the line with equation $x = 3$, leaving your answer in terms of e. (AQA)

23 The first four terms of three series A, B and C are given below. One series is an arithmetic series, one is a geometric series and one is neither.

A: $8 + 4 + 2 + 1 + \ldots$
B: $1 + 4 + 8 + 13 + \ldots$
C: $23 + 20 + 17 + 14 + \ldots$

a State the value of the common difference and calculate the 21st term of the arithmetic series.

b Find the sum to infinity of the geometric series. (AQA)

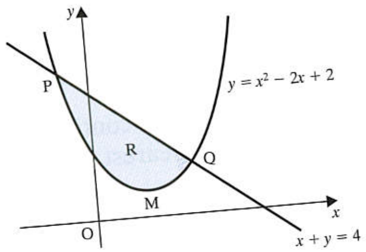
24 $f(x) \equiv \dfrac{(2\sqrt{x}+3)^2}{x},\ x > 0$

a Show that $f(x)$ can be expressed as $A + Bx^{-\frac{1}{2}} + Cx^{-1}$, giving the values of the constants A, B and C.

b Find $\int f(x)\,dx$.

c Find the area of the finite region bounded by the curve with equation $y = f(x)$ and the lines with equations $x = 4$, $x = 9$ and $y = 0$, giving your answer in terms of natural logarithms. (Edexcel)

25 The diagram below shows sketches of the line with equation $x + y = 4$ and the curve with equation $y = x^2 - 2x + 2$ intersecting at points P and Q. The minimum point of the curve is M. The shaded region R is bounded by the line and the curve.



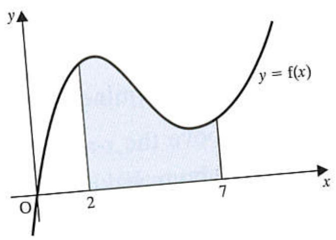
a Show that the coordinates of M are $(1, 1)$.

b Find the coordinates of the points P and Q.

c Prove that the triangle PMQ is right-angled and hence show that the area of the triangle PMQ is 3 square units.

d Show that the area of the region R is $1\frac{1}{2}$ times that of the triangle PMQ. (AQA)

26



For some function f, part of the graph of $y = f(x)$ is illustrated above. It is given that the shaded region has an area equal to 20 square units.

a Sketch the graph of $y = f(x) + 3$ and calculate

$$\int_2^7 (f(x) + 3)\,dx$$

b Find the value of the constant k for which

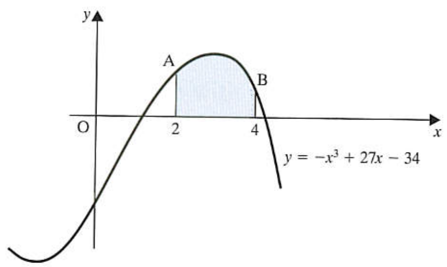
$$\int_2^7 (f(x) + k)\,dx = 0$$

(AQA)

27 The nth term of a sequence is ar^{n-1}, where a and r are constants. The first term is 3 and the second term is $-\frac{3}{4}$. Find the values of a and r.

Hence find the value of $\sum_{n=1}^{\infty} ar^{n-1}$. (OCR)

28



The figure shows a sketch of part of the curve with equation $y = f(x)$ where $f(x) = -x^3 + 27x - 34$.

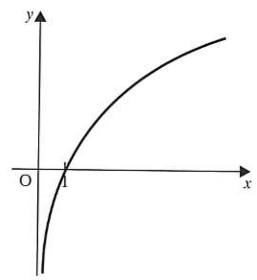
a Find $\int f(x)\,dx$.

The lines $x = 2$ and $x = 4$ meet the curve at points A and B as shown.

b Find the area of the finite region bounded by the curve and the lines $x = 2$, $x = 4$ and $y = 0$.

c Find the area of the finite region bounded by the curve and the straight line AB. (Edexcel)

29 The figure below shows the graph of $y = \ln x$ for $x > 0$.



Use this diagram to sketch the graphs of

a $y = |\ln x|$ for $x > 0$;　　**b** $y = \ln|x|$ for all real x, $x \neq 0$. (OCR)

30 Solve the equation $|x| = |2x + 1|$ (OCR)

31 At the beginning of 1990, an investor decided to invest £6000 in a Personal Equity Plan (PEP), believing that the value of the investment should increase, on average, by 6% each year. Show that, if this percentage rate of increase is in fact maintained for 10 years, the value of the original investment will be about £10 745.

The investor added a further £6000 to the PEP at the beginning of each year between 1991 and 1995 inclusive. Assuming that the 6% annual rate of increase continues to apply, show that the total value, in £, of the PEP at the beginning of the year 2000 may be written as

$$6000\sum_{r=5}^{10}(1.06)^r$$

and evaluate this, correct to the nearest £. (OCR)

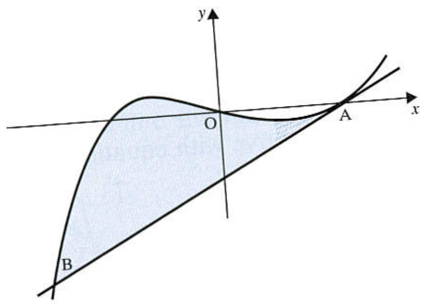
32 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 2n^2$

i Write down the value of u_3.

ii Express $u_{n+1} - u_n$ in terms of n, simplifying your answer.

iii The differences between successive terms of the sequence form an arithmetic progression. For this arithmetic progression, state its first term and its common difference, and find the sum of its first 1000 terms. (OCR)

33



The diagram shows a sketch of the graph of the curve $y = x^3 - x$ together with the tangent to the curve at the point $A(1, 0)$.

i Use differentiation to find the equation of the tangent to the curve at A, and verify that the point B where the tangent cuts the curve again has coordinates $(-2, -6)$.

ii Use integration to find the area of the region bounded by the curve and the tangent (shaded in the diagram), giving your answer as a fraction in its lowest terms. (OCR)

34 At the start of a particular year, Mrs Brown made a single investment of £2000. At the end of that year and at the end of each subsequent year the value of her investment was 10% greater than its value at the start of the year.
Find, to the nearest £, the value of Mrs Brown's investment at the end of

a the fifth year **b** the tenth year.

Mrs Chan decided to invest £2000 at the start of each year with the same broker and at a fixed rate of interest of 10% per annum.

c Write down the first three terms of a series whose sum is the total value of Mrs Chan's annual investment at the end of the 12 years.

d Hence determine the value, to the nearest £, of Mrs Chan's investment at the end of 12 years. (Edexcel)

35 All the integers which are exactly divisible by 3 and lie between 1 and 100 form a series. Find

a the number of terms in the series

b the sum of the terms in the series. (Edexcel)

36 It is given that $y = x^{\frac{3}{2}} + \dfrac{48}{x}$, $x > 0$.

a Find the value of x and the value of y when $\dfrac{dy}{dx} = 0$.

b Show that the value of y which you found in part (a) is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \dfrac{48}{x}$, the lines $x = 1$, $x = 4$ and the x-axis.

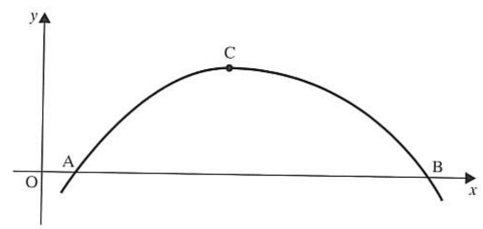
c Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p, q and r are to be found. (Edexcel)

37 A pump is used to extract air from a bottle. The first operation of the pump extracts $56\,\text{cm}^3$ of air and subsequent extractions follow a geometric progression. The third operation of the pump extracts $31.5\,\text{cm}^3$ of air.

a Determine the common ratio of the geometric progression and calculate the total amount of air that could be extracted from the bottle, if the pump were to extract air indefinitely.

b After how many operations of the pump does the total amount of air extracted from the bottle first exceed $220\,\text{cm}^3$? (AQA)

38



The function f is defined for positive real values of x by

$$f(x) = 12 \ln x - x^{\frac{3}{2}}.$$

The figure shows a sketch of the curve with equation $y = f(x)$. The curve crosses the x-axis at the points A and B. The gradient of the curve is zero at the point C.

a By calculation, show that the value of x at the point A lies between 1.1 and 1.2.

The value of x at the point B lies in the interval $(n, n+1)$, where n is an integer.

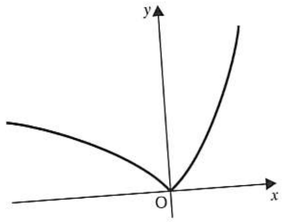
b Determine the value of n.

c Show that $x = 4$ at the point C and hence find the greatest positive value of $f(x)$, giving your answer to 2 decimal places.

d Write down the set of values of x for which $f(x)$ is an increasing function of x. (Edexcel)

39 The ninth term of an arithmetic progression is 52 and the sum of the first twelve terms is 414. Find the first term and the common difference. (AQA)

40



The diagram shows the graph of $y = |f(x)|$, for a certain function f with domain $\mathbb{R}$. Sketch, on separate diagrams, two possibilities for the graph of $y = f(x)$. (OCR)

41 An employer offers the following schemes of salary payments over a five-year period:

Scheme X: 60 monthly payments, starting with £1000 and increasing by £6 each month [£1000, £1006, £1012, . . .];

Scheme Y: 5 annual payments, starting with £12 000 and increasing by £d each year [£12 000, £(12 000 + d), . . .].

a Over the complete five-year period, find the total salary payable under Scheme X.

b Find the value of d which gives the same total salary for both schemes over the complete five-year period. (Edexcel)

42 An athlete plans a training schedule which involves running 20 km in the first week of training; in each subsequent week the distance is to be increased by 10% over the previous week. Write down an expression for the distance to be covered in the nth week according to the schedule, and find in which week the athlete would first cover more than 100 km. (OCR)

43 i The tenth term of an arithmetic progression is 36, and the sum of the first ten terms is 180. Find the first term and the common difference.

ii Evaluate $\sum_{r=1}^{1000} (3r - 1)$. (OCR)

44 The functions f and g are defined by

$$f : x \mapsto 3x - 1, \quad x \in \mathbb{R},$$
$$g : x \mapsto x^2 + 1, \quad x \in \mathbb{R},$$

a Find the range of g.

b Calculate the value of gf(2).

c Determine the values of x for which $gf(x) = fg(x)$,

d Determine the values of x for which $|f(x)| = 8$. (Edexcel)

45 The functions f and g are defined by

$$f : x \mapsto x^2 - 10, \quad x \in \mathbb{R},$$
$$g : x \mapsto |x - 2|, \quad x \in \mathbb{R}$$

a Show that $f \circ f : x \mapsto x^4 - 20x^2 + 90, \; x \in \mathbb{R}$. Find all the values of x for which $f \circ f(x) = 26$.

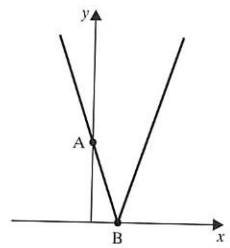
b Show that $g \circ f(x) = |x^2 - 12|$. Sketch a graph of $g \circ f$. Hence, or otherwise, solve the equation $g \circ f(x) = x$. (AQA)

46 A small ball is dropped from a height of 1 m onto a horizontal floor. Each time the ball strikes the floor it rebounds to $\frac{3}{5}$ of the height from which it has just fallen.

a Show that, when the ball strikes the floor for the third time, it has travelled a distance 2.92 m.

b Show that the total distance travelled by the ball cannot exceed 4 m. (Edexcel)

47



The figure shows a sketch of the graph $y = |3x - 2|$.

i State the coordinates of the points labelled A and B.

ii Make a copy of the figure, and shade in the area represented by

$$\int_0^2 |3x - 2| \, dx.$$

iii Evaluate this area. (OCR)

48 Solve the inequality $f|x + 1| < |x - 2|$. (OCR)

3 $\dfrac{d}{dx}\ln\left(\dfrac{x+1}{2x}\right)$ is

A $\dfrac{1}{2}$ B $\dfrac{1}{x+1}-\dfrac{1}{2x}$

C $\dfrac{2x}{x+1}$ D $\dfrac{1}{x+1}-\dfrac{1}{x}$

E $-\dfrac{1}{x(x+1)}$

4 $\dfrac{d}{dx}a^x$ is

A xa^{x-1} B a^x

C $x\ln a$ D $a^x\ln a$

5 If $x = \cos\theta$ and $y = \cos\theta + \sin\theta$, $\dfrac{dy}{dx}$ is

A $1-\cot\theta$ B $1-\tan\theta$

C $\cot\theta - 1$ D $\cot\theta + 1$

6 The greatest value of $5\cos\theta - 4\sin\theta$ is

A 3 B 1 C $\sqrt{41}$ D ± 5

7 $3\cos\theta - 4\sin\theta =$

A $5\cos(\theta+\alpha)$ where $\tan\alpha = \frac{3}{4}$

B $5\sin(\alpha-\theta)$ where $\tan\alpha = \frac{3}{4}$

C $5\cos(\theta+\alpha)$ where $\tan\alpha = \frac{4}{3}$

D $-5\cos(\theta-\alpha)$ where $\tan\alpha = \frac{4}{3}$

8 If $y = \ln(\ln x)$ and $x > 1$ then

A $\dfrac{dy}{dx} = \dfrac{1}{\ln x}$ C $\dfrac{dy}{dx} = \dfrac{1}{x\ln x}$

B $e^y = \ln x$ D $y = \ln x^2$

9 Given that $x = \cos^2\theta$ and $y = \sin^2\theta$,

A $x^2+y^2 = 1$ C $0 \leqslant y \leqslant 1$

B $\dfrac{dy}{dx} = \tan\theta$ D $y = x - \frac{1}{2}\pi$

10 $2\cos(2\theta - 60°) \equiv$

A $\cos 2\theta - \sqrt{3}\sin 2\theta$

B $\sin 2\theta - \sqrt{3}\cos 2\theta$

C $\cos 2\theta + 2\sqrt{3}\cos\theta\sin\theta$

D $\cos 2\theta + \sqrt{3}\sin 2\theta$

11 $\tan\theta = 0.8 \Rightarrow \tan 2\theta =$

A 0.4 C 1.6

B 40/9 D $4.\dot{4}$

In Questions 12 to 19 a single statement is made. Write T if it is true and F if it is false.

12 $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}\times\dfrac{dv}{dx}$

13 $\dfrac{d}{dx}(x^2y^2) = 2xy^2 + 2x^2y$

14 Given that $y = \ln x^2$ and x increases by δx then $\delta y \approx \left(\dfrac{1}{x^2}\right)(\delta x)$.

15 When $y = \cos 2\theta$ and $x = \sin\theta$, $\dfrac{dy}{dx} = -4x\sqrt{1-x^2}$.

16 If $y = f(t)$ and $x = g(t)$ then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$.

17 $y = e^{3x} \iff \dfrac{dy}{dx} = 3e^{3x}$

18 $y = 5\sin\theta + 12\cos\theta \iff y = 13\sin(\theta+\alpha)$ where $\tan\alpha = \frac{12}{5}$

19 $\sin 2\theta = 0.6 \Rightarrow \sin\theta = 0.3$.

Examination Questions E

1 A student is asked to express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$. She writes

$$3\sin\theta + 4\cos\theta \equiv 5\sin\left(\theta + \frac{\pi}{3}\right)$$

Determine, with a reason, whether she is correct. (AQA)

2 Find the equation of the tangent to the curve $y = (4x+3)^5$ at the point $(-\frac{1}{2}, 1)$, giving your answer in the form $y = mx + c$. (OCR)

3 Differentiate each of the following functions with respect to x.

a $\dfrac{1}{\sqrt{x}}$ b e^{-x} c $x^2\cos(2x)$ (OCR)

4 The parametric equations of a curve are $x = \ln t$, $y = t + t^2$, where $t > 0$.

Express $\dfrac{dy}{dx}$ in terms of t, simplifying your answer. (OCR)

5 The function f is defined by $f: x \mapsto 5 + 3x^2 - 36\ln(x-1)$ and has domain $x \geqslant 2$.

a Determine the value of x for which f is stationary.

b The second derivative of $f(x)$ is $f''(x)$. Find the range of f''. (AQA)

6 Find the four solutions of the equation $\sin 2\theta = \cos^2\theta$ in the interval $0° < \theta < 360°$. Give each of these solutions correct to the nearest degree. (AQA)

7 Express $\sin 4\theta$ in terms of $\sin 2\theta$ and $\cos 2\theta$, and hence express $\dfrac{\sin 4\theta}{\sin\theta}$ in terms of $\cos\theta$ only. (OCR)

8 Find the x-coordinates of the stationary points of $y = x^3e^{-kx}$, where k is a positive constant. (OCR)

9 Find all values of θ, for $0 \leqslant \theta \leqslant 180°$, which satisfy the equation $\cos 2\theta = \cos\theta$. (OCR)

10 Given that

$$3\cos x - 4\sin x \equiv R\cos(x+\alpha),$$

where $R > 0$ and $0° < \alpha < 90°$, find the values of R and α, giving the value of α correct to two decimal places.

Hence solve the equation

$$3\cos 2\theta - 4\sin 2\theta = 2,$$

for $0° < \theta < 360°$, giving your answers correct to one decimal place. (OCR)

11 The curve C has parametric equations

$$x = 4\cos 2t,\ y = 3\sin t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

A is the point $(2, 1\frac{1}{2})$ and lies on C.

a Find the value of t at the point A.

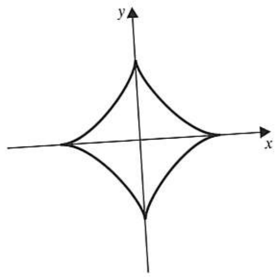
b Find $\frac{dy}{dx}$ in terms of t.

c Show that an equation of the normal to C at A is $6y - 16x + 23 = 0$.

The normal at A cuts C again at the point B.

d Find the y-coordinate of the point B. (Edexcel)

12



The curve shown above is called an astroid and is defined by the parametric equations

$$x = 8\cos^3\theta,\ \ y = 8\sin^3\theta \text{ for } 0 \leqslant \theta \leqslant 2\pi.$$

a By obtaining expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$, show clearly that the gradient of the curve is given by

$$\frac{dy}{dx} = -\tan\theta.$$

The astroid above can also be defined by the Cartesian equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k.$$

b Find the value of the constant k by first finding values of x and y for a particular value of θ.

c Use implicit differentiation to find an expression for $\frac{dy}{dx}$ in terms of x and y. (AQA)

13 Find all values of θ in the range 0° to 360° satisfying

a $2\sin 2\theta = \sin\theta$ **b** $3\sec^2\theta + 5\tan\theta - 5 = 0$ (NEAB)

14 Differentiate the following with respect to x, simplifying your answers where possible.

a $x^2 \ln x$ **b** $\frac{3x^2 - 5}{2x^2 + 7}$ **c** $(x^3 + 5)^{10}$ (NEAB)

15 Given that

$$a\cos\theta + b\sin\theta = r\cos(\theta - \alpha) \text{ where } r > 0,$$

show that

$$r = \sqrt{a^2 + b^2}$$

and find an expression for $\tan\alpha$ in terms of a and b.

Hence find all values of θ between 0° and 360° satisfying the equation

$$2\cos\theta + 3\sin\theta = 1.$$

Give your answers correct to the nearest degree. (AQA)

16 a Write down the exact value for $\cos 30°$.

b Use an appropriate double angle formula to write $\cos 15°$ as $\sqrt{a + \sqrt{b}}$, where a and b are rational. (OCR)

17 Write down the derivative of $\tan 2x$ with respect to x. (OCR)

18 Differentiate the following functions with respect to x, simplifying your answers as far as possible.

a $x^2\sin 2x$ **b** $\frac{x}{\sqrt{x^2 - 1}}$ (AQA)

19 The parametric equations of a curve are $x = e^{2t} - 5t,\ y = e^{2t} - 2t$. Find $\frac{dy}{dx}$ in terms of t.

Find the exact value of t at the point on the curve where the gradient is 2. (OCR)

20 a Let $u = \sin v$, write down $\frac{du}{dv}$ as a function of v.

b Hence, or otherwise, obtain $\frac{dv}{du}$ in terms of u. (OCR)

21 a Express the function $2\sin x° + \cos x°$ in the form $R\sin(x + \alpha)°$, stating the values of R and α. Using these values, write down the coordinates of the maximum turning point on the graph of

$$2\sin x° + \cos x° \text{ for } 0 \leqslant x \leqslant 90.$$

b Express $3\cos 2x + \sin x$ in terms of $\sin x$. Hence calculate all of the values of x between 0 and 360 which satisfy the equation

$$3\cos 2x° + \sin x° = 1.$$

(AQA)

22 Express $2\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the values of R and α in an exact form.

Hence, or otherwise, show that one of the acute angles θ satisfying the equation $2\cos\theta + 2\sin\theta = \sqrt{6}$ is $\frac{5}{12}\pi$, and find the other acute angle. (OCR)

23 Differentiate $x^2\ln x$ with respect to x. (OCR)

24 Given that θ lies in the interval $0° < \theta < 45°$,

a show that $\cot\theta + \tan\theta = \frac{2}{\sin 2\theta}$

b find the values of θ for which $\cot\theta + \tan\theta > 4$. (AQA)

25 A curve C is defined by the parametric equations

$$x = e^{t} + t, \quad y = e^{-t} + t.$$

Express $\frac{dy}{dx}$ in terms of t. (AQA)

26 The curve C has parametric equations

$$x = t^3, \quad y = t^2, \quad t > 0.$$

a Find an equation of the tangent to C at A(1, 1).

Given that the line l with equation $3y - 2x + 4 = 0$ cuts the curve C at point B,

b find the coordinates of B,

c prove that the line l only cuts C at the point B. (Edexcel)

27

$$8\cos x^\circ - 15\sin x^\circ \equiv R\cos(x + A)^\circ, \quad 0 \leqslant A \leqslant 90, \quad R > 0$$

Find

a the value of R and, to one decimal place, the value of A,

b the maximum value of $8\cos x^\circ - 15\sin x^\circ$, and the smallest positive value of x for which this occurs,

c the two smallest values of x for which $8\cos x^\circ - 15\sin x^\circ = 6$. (Edexcel)

28 A curve C is defined by the equation

$$x^2 - 2xy + 3y^2 = 9$$

i Show that the point P(3, 2) lies on the curve.

ii Show that at P

$$\frac{dy}{dx} = -\frac{1}{3}.$$

iii The point Q$(3 + h, 2 + k)$ also lies on the curve C and is close to P. Using the result of part ii, write down an approximate expression for k in terms of h. (AQA)

29 Differentiate with respect to x, **a** $\frac{\sin x}{x}$, $x > 0$ **b** $\ln\left(\frac{1}{x^2 + 9}\right)$

Given that $y = x^x$, $x > 0$, $y > 0$, by taking logarithms

c show that $\frac{dy}{dx} = x^x(1 + \ln x)$. (Edexcel)

30 Given that $y = \cos 2x + \sin x$, $0 < x < 2\pi$, and x is in radians,

a find, in terms of π, the values of x for which $y = 0$.

b Find, to 2 decimal places, the values of x for which $\frac{dy}{dx} = 0$. (Edexcel)

31 Starting from the identity for $\cos(A + B)$, prove that

$$\cos 2x = 1 - 2\sin^2 x.$$

Find, in radians to 2 decimal places, the values of x in the interval $0 \leqslant x < 2\pi$ for which

a $2\cos 2x + 1 = \sin x$ **b** $2\cos x + 1 = \sin \tfrac{1}{2}x$ (Edexcel)

32 a Express $5\sin x + 12\cos x$ in the form $R\sin(x + \theta)$ where $R > 0$, and $0^\circ < \theta < 90^\circ$.

b Hence, or otherwise, find the maximum and minimum values of $f(x)$ where

$$f(x) = \frac{30}{5\sin x + 12\cos x + 17}$$

State also the values of x, in the range $0^\circ < x < 360^\circ$, at which they occur. (OCR)

33 a Find the equation of the tangent to the curve $y = e^{-x}$ at the point P(t, e^{-t}).

b The tangent at P meets the x- and y-axes at the points Q and R respectively. Show that A, the area of triangle OQR, where O is the origin, is given by

$$A = \tfrac{1}{2}(t + 1)^2 e^{-t}.$$

c Find, showing your working, the stationary values of A and determine their nature. (WJEC)

34 A curve is defined parametrically by $x = (2t - 1)$, $y = t^3$ and P is the point on the curve where $t = 2$.

a Obtain an expression for $\frac{dy}{dx}$ in terms of t and calculate the gradient of the curve at P.

b Determine a cartesian equation of the curve, expressing your answer in the form $y = f(x)$.

c Sketch the curve, showing clearly the values of the intercepts on the axes.

d Write down the equation of the tangent to the curve at P. This tangent intersects the curve again at Q, with parameter q. Show that $q^3 = 12q - 16$. Hence determine the coordinates of the point Q.

e Prove that the normal to the curve at P does not intersect the curve at any other point. (AQA)

35 i Differentiate with respect to x **a** $e^{\sqrt{x}}$, **b** $\frac{1 + x^2}{\ln x}$

ii A curve is given by the parametric equations

$$x = \cos 2t, \quad y = 4\sin^3 t, \quad 0 \leqslant t \leqslant \frac{\pi}{4}.$$

a Show that $\frac{dy}{dx} = -3\sin t$.

b Find an equation of the normal to the curve at the point where $t = \frac{\pi}{6}$. (Edexcel)

36 The volume, V, of a sphere of radius r is given by $V = \tfrac{4}{3}\pi r^3$.

a Obtain an expression for $\frac{dV}{dr}$.

b A balloon when almost fully inflated can be modelled by a sphere. When the radius of the balloon is 15 cm, it is observed that the rate of increase of the radius is 0.1 cm s^{-1}. Find, to two significant figures, the rate of increase of the volume at this time. (AQA)

37 The volume of liquid, $V\,\text{cm}^3$, in a container when the depth is x cm ($x > 0$) is given by

$$V = \frac{30\sqrt{x}}{(x+9)}$$

The container has height h cm.

a Given that $x = h$ when $\frac{dV}{dx} = 0$, find the value of h, and hence determine the maximum capacity of the container.

b Calculate the rate of change of volume when the depth is 1 cm and increasing at a rate of $0.02\,\text{cm s}^{-1}$, giving your answer in $\text{cm}^3\,\text{s}^{-1}$. (AQA)

38 In a simple model of the tides in a harbour, the depth, y m, is given by

$$y = 20 + 5\cos t,$$

where t is the time measured in suitable units. For $0 \leqslant t \leqslant 4\pi$, write down the maximum and minimum values of y and the values of t when they occur.

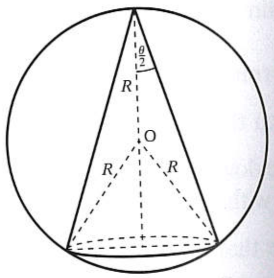
In a more refined model, y is given by

$$y = 20 + \{5 + \cos(\tfrac{1}{2}t)\}\cos t.$$

Using this model, show that the values of t when y is stationary satisfy the equation

$$\sin(\tfrac{1}{2}t)\{6\cos^2(\tfrac{1}{2}t) + 20\cos(\tfrac{1}{2}t) - 1\} = 0$$ (OCR)

39 A packaging designer places a right circular cone inside a hollow sphere of fixed radius R and centre O, as shown in the figure, where the vertical angle of the cone is θ.



a Prove that the volume V of the cone is given by $V = \frac{\pi}{3}R^3(1 + \cos\theta)\sin^2\theta$.

The designer needs to find the maximum volume of the cone when R is fixed and θ varies.

b Show, by differentiation, that $\arccos\frac{1}{3}$ is the value of θ for which V is a maximum.

c Find the maximum value of V, in terms of R. (Edexcel)

40 A population P is growing at the rate of 9% each year and at time t years may be approximated by the formula

$$P = P_0(1.09)^t, \quad t \geqslant 0,$$

where P is regarded as a continuous function of t and P_0 is the starting population at time $t = 0$.

a Find an expression for t in terms of P and P_0.

b Find the time T years when the population has doubled from its value at $t = 0$, giving your answer to 3 significant figures.

c Find, as a multiple of P_0, the rate of change of population $\frac{dP}{dt}$ at time $t = T$. (Edexcel)

41 Find all the values of θ, such that $0 \leqslant \theta \leqslant 180°$, which satisfy the equation $2\sin 2\theta = \tan\theta$. (OCR)

42 The curve with equation $ky = a^x$ passes through the points with coordinates (7, 12) and (12, 7). Find, to 2 significant figures, the values of the constants k and a.

Using your values of k and a, find the value of $\frac{dy}{dx}$ at $x = 20$, giving your answer to 1 decimal place. (Edexcel)

43 A speaker uses an amplifier to carry her words to members of the audience x metres away. The power output, P watts, is given by the formula $P = 0.0004x^2$.

i To increase the distance by a small amount δx metre, the output must be increased by δP watt. Find an approximate expression for δP in terms of x and δx.

ii Show that $\frac{\delta P}{P} \approx 2\frac{\delta x}{x}$.

iii If the power output of the amplifier is increased by 2%, by what percentage approximately is the distance her voice will carry increased? (OCR)

44 The equation of a closed curve is $(x+y)^2 + 2(x-y)^2 = 24$.

i Show, by using differentiation, that the gradient at the point (x, y) on the curve may be expressed in the form $\frac{3x - y}{x - 3y}$.

ii Find the coordinates of all the points on the curve at which the tangent is parallel to either the x-axis or the y-axis.

iii Find the exact coordinates of all the points at which the curve crosses the axes, and the gradient of the curve at each of these points. (OCR)

45 A forest fire spreads so that the number of hectares burnt after t hours is given by

$$h = 30(1.65)^t$$

i By what constant factor is the burnt area multiplied from time $t = N$ to time $t = N + 1$? Express this as a percentage increase.

ii 1.65 can be written as e^K. Find the value of K.

iii Hence show that $\frac{dh}{dt} = 15e^{Kt}$.

iv This shows that $\frac{dh}{dt}$ is proportional to h. Find the constant of proportionality. (OCR)

14 Which of the following differential equations can be solved by separating the variables?

A $x\dfrac{dy}{dx} = y + x$

B $xy\dfrac{dy}{dx} = x + 1$

C $e^{x+y} = y\dfrac{dy}{dx}$

D $x + \dfrac{dy}{dx} = \ln y$

15 $\displaystyle\int_1^2 x e^x \, dx$

A is a definite integral

B is equal to $xe^x - e^x$

C is equal to $\left[\frac{1}{2}e^{x^2}\right]_1^2$

D can be integrated by parts.

In Questions 16 to 21 a single statement is made. Write T if it is true, F if it is false.

16 $\displaystyle\sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\,dx$

17 $x^2 + 2y^2 = 1$ is the equation of a circle.

18 $\displaystyle\int \tan x\,dx = \sec^2 x + K$

19 $\displaystyle\int_0^a f(y)\,dy = \lim_{\delta y \to 0} \sum_{y=0}^{y=a} f(y)\,\delta y$

20 $\left[f(x)\right]_0^a = f(a) - 0$

21 $x^2 + y^2 - 2x - 4y + 6 = 0$ is the equation of a circle.

Examination Questions F

1 The polynomial $x^3 + 2x^2 + ax + b$

i is divisible by $(x - 1)$ **ii** leaves a remainder 12 when divided by $(x - 2)$.

Find the values of a and b.

(AQA)

2 Find $\displaystyle\int (2 + e^{-x})\,dx$.

(OCR)

3 Sketch the curve defined by the equations

$$x = 3a\cos t,\ y = 2a\sin t,$$

where a is a positive constant, for $0 \leqslant t \leqslant 2\pi$.

(AQA)

4 Use integration by parts to show that $\displaystyle\int_2^4 x\ln x\,dx = 7\ln 4 - 3$

(Edexcel)

5 $$\frac{5x - 1}{(1 + x)(1 + x + 3x^2)} = \frac{Ax + B}{1 + x + 3x^2} + \frac{C}{1 + x}$$

Find the values of the constants A, B and C.

(Edexcel)

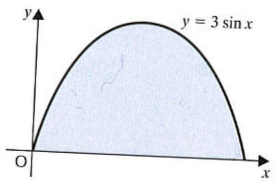
6 $$f(x) = \frac{x - 8}{(x + 2)(2x - 1)}$$

a Express $f(x)$ in partial fractions. **b** Find $f'(0)$.

(Edexcel)

7 Using the substitution $2x = \sin\theta$, or otherwise, find the exact value of $\displaystyle\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1 - 4x^2}}\,dx$. (AQA)

8



The graph of $y = 3\sin x$ is sketched above.
Write down an expression for the shaded area. Hence find the value of the shaded area. (AQA)

9 Find **i** $\displaystyle\int \frac{1}{2x + 1}\,dx$ **ii** $\displaystyle\int (2x + 1)^7\,dx$

(OCR)

10 The points $(5, 5)$ and $(-3, -1)$ are the ends of a diameter of the circle C with centre A. Write down the coordinates of A and show that the equation of C is

$$x^2 + y^2 - 2x - 4y - 20 = 0.$$

The line L with equation $y = 3x - 16$ meets C at the points P and Q.
Show that the x-coordinates of P and Q satisfy the equation

$$x^2 - 11x + 30 = 0$$

Hence find the coordinates of P and Q.

(WJEC)

11 The polynomial

$$p(x) = x^3 + cx^2 + 7x + d$$

has a factor of $x + 2$, and leaves a remainder of 3 when divided by $x - 1$.

a Determine the value of each of the constants c and d.

b Find the exact values of the three roots of the equation $p(x) = 0$. (AQA)

12 Express $\dfrac{x}{(x+1)(x+2)^2}$ in partial fractions. (AQA)

13 a Show that $2x - 1$ is a factor of $8x^4 - 4x^3 + 14x^2 - 3x - 2$.

b Find the remainder when $8x^4 - 4x^3 + 14x^2 - 3x - 2$ is divided by $x + 1$. (OCR)

14 a Find the radius and the coordinates of the centre of the circle with equation

$$x^2 + y^2 - 2x - 8y = 0.$$

b Determine, by calculation, whether the point $(2.9, 1.7)$ lies inside or outside the circle. (Edexcel)

15 Evaluate, in terms of e, $\displaystyle\int_0^1 (e^{2x} - 1)^2\,dx$ (Edexcel)

16 $f(n) \equiv n^2 + n + 1$, where n is a positive integer.
Classify the following statements about $f(n)$ as true or false. If a statement is true, prove it; if it is false, provide a counter-example.

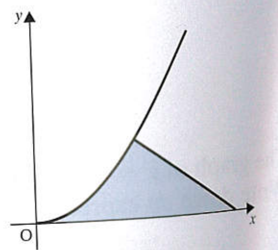
a $f(n)$ is always a prime number.

b $f(n)$ is always an odd number. (Edexcel)

17 The curve shown has parametric equations

$$x = t^2,\ y = t^3,$$

where $t \geqslant 0$ is a parameter.
Also shown is part of the normal to the curve at the point where $t = 1$.



a Find an equation of this normal.

b Find the area of the finite region bounded by the curve, the x-axis and this normal. (Edexcel)

18 Use the identity $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$ to prove that $\sin^2\theta \equiv \frac{1}{2}(1 - \cos 2\theta)$.

The finite region R is bounded by the curve with equation $y = \sin^2 2x$, the lines $x = \dfrac{\pi}{8}$,

$x = \dfrac{\pi}{4}$ and $y = 0$.

Find the area of R giving your answer in the form $p + q\pi$, where p and q are numbers to be found. (Edexcel)

19 Find $\displaystyle\int x\cos 2x\,dx$. (OCR)

20 Use the substitution $u = 3x - 1$ to express $\displaystyle\int x(3x-1)^4\,dx$ as an integral in terms of u.

Hence, or otherwise, find $\displaystyle\int x(3x-1)^4\,dx$, giving your answer in terms of u. (OCR)

21 A circle C has equation

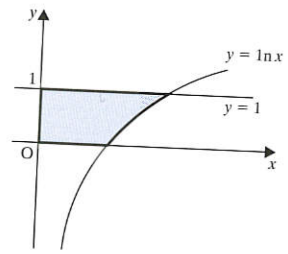
$$x^2 + y^2 - 2x - 8y - 8 = 0.$$

a Find the coordinates of the centre of C.

b Find the radius of C.

c Find the equation of the tangent to C at the point $(5, 1)$. (WJEC)

22



The region R in the first quadrant is bounded by the curve $y = \ln x$, the x-axis, the y-axis and the line $y = 1$, as shown in the diagram. Show that the volume of the solid formed when R is rotated completely about the y-axis is $\frac{1}{2}\pi(e^2 - 1)$. (OCR)

23 A class is asked to prove the result that for all positive integers n, if

$$S_n = 1 + 3 + 5 + \ldots + (2n - 1),\ \text{then } S_n = n^2.$$

a The following is a student's attempt at a proof.

$S_1 = 1 \qquad\qquad\quad = 1^2$
$S_2 = 1 + 3 = 4 \qquad = 2^2$
$S_3 = 1 + 3 + 5 = 9 = 3^2$
etc.

so the formula is true for all positive integers n.
Why is this student's argument incorrect?

b Another student begins to prove the result in the following way.

$S_n = 1 \quad + 3 \quad + 5 \quad + \ldots + (2n - 1).$
Also, $S_n = (2n-1) + (2n-3) + (2n-5) + \ldots + 1.$
Adding each term, we have ...

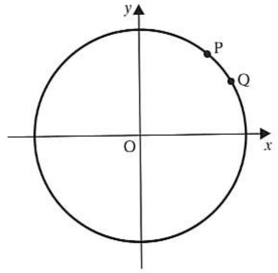
Complete this student's proof correctly to show that $S_n = n^2$ for all positive integers n. (AQA)

24 Determine the coordinates of the centre C and the radius of the circle with equation

$$x^2 + y^2 + 4x - 10y + 13 = 0.$$

Find the distance from the point P(2, 3) to the centre of the circle.
Hence find the length of the tangents from P to the circle. (AQA)

25



The diagram shows the circle with equation $x^2 + y^2 = 25$. Find the length of the minor arc between the points P(3, 4) and Q(4, 3), giving your answer correct to 3 significant figures. (OCR)

26 Express $\dfrac{3}{(2x+1)(x-1)}$ in partial fractions.

Hence find the exact value of $\displaystyle\int_2^3 \frac{3}{(2x+1)(x-1)}\,dx$ giving your answer as a single logarithm. (OCR)

27 $f(x) \equiv \dfrac{5x^2 - 8x + 1}{2x(x-1)^2} \equiv \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

a Find the values of the constants A, B and C.

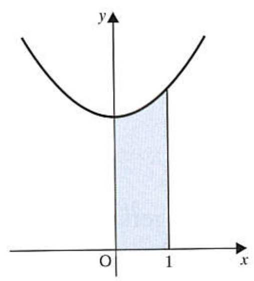
b Hence find $\int f(x)\,dx$.

c Hence show that

$$\int_4^9 f(x)\,dx = \ln\left(\tfrac{32}{3}\right) - \tfrac{5}{24}.$$

(Edexcel)

28



The figure shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x-axis. This region is rotated through 360° about the y-axis. Find the volume generated. (Edexcel)

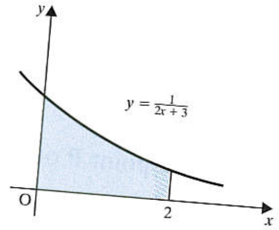
29 a i Find $\displaystyle\int \frac{1}{x(x+1)}\,dx,\ x > 0$.

Using the substitution $u = e^x$ and the answer to **i**, or otherwise,

ii find $\displaystyle\int \frac{1}{1+e^x}\,dx$. (Edexcel)

b Use integration by parts to find $\int x^2 \sin x\,dx$.

30 a



Calculate the volume of the solid formed when the area between $y = \dfrac{1}{2x+3}$, the axes and the line $x = 2$ is rotated through an angle of 2π radians about the x-axis.

b Use the substitution $u = 2x + 3$ to find $\displaystyle\int \frac{x}{(2x+3)^2}\,dx$. (AQA)

31 Find

a $\int \sin 2y\,dy$,

b $\int x e^x\,dx$.

c Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{xe^x}{\sin y \cos y},\quad 0 < y < \frac{\pi}{2}.$$

(Edexcel)

32 a Obtain the general solution of the differential equation

$$\frac{dy}{dx} = xy^2,\ y > 0$$

b Given also that $y = 1$ at $x = 1$ show that

$$y = \frac{2}{3 - x^2},\quad -\sqrt{3} < x < \sqrt{3},$$

is a particular solution of the differential equation.

The curve C has equation $y = \dfrac{2}{3-x^2}$, $x \neq -\sqrt{3}$, $x \neq \sqrt{3}$.

c Write down the gradient of C at the point (1, 1).

d Deduce that the line which is a tangent to C at the point (1, 1) has equation $y = x$.

e Find the coordinates of the point where the line $y = x$ again meets the curve C. (Edexcel)

Sketch on the same axes the curves with equations $y = 1 + x^2$ and $y = 9 - x^2$ and determine their points of intersection.
The finite region bounded by the two curves is rotated through π radians about the y-axis.
Calculate, in terms of π, the volume of the solid of revolution. (AQA)

At time t hours the rate of decay of the mass of a radioactive substance is proportional to the mass x kg of the substance at that time. At time $t = 0$ the mass of the substance is A kg.

a By forming and integrating a differential equation, show that $x = Ae^{-kt}$, where k is a constant.
It is observed that $x = \frac{1}{3}A$ at time $t = 10$.

b Find the value of t when $x = \frac{1}{2}A$, giving your answer to 2 decimal places. (Edexcel)

35 The curve C is given by the equations

$$x = 2t, y = t^2,$$

where t is a parameter.

a Find an equation of the normal to C at the point P on C where $t = 3$.

The normal meets the y-axis at the point B. The finite region R is bounded by the part of the curve C between the origin O and P, and the lines OB and BP.

b Show the region R, together with its boundaries, in a sketch.

The region R is rotated through 2π about the y-axis to form a solid S.

c Using integration, and explaining each step in your method, find the volume of S, giving your answer in terms of π. (Edexcel)

36 Showing your method clearly in each case, find

a $\int \sin^2 x \cos x \, dx$

b $\int x \ln x \, dx$

c Use the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$, to find $\int \frac{x}{\sqrt{x+1}} dx$.

d Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$. (Edexcel)

37 The rate, in $cm^3 s^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, V cm^3, in the sump at that instant. At time $t = 0$, $V = A$.

a By forming and integrating a differential equation, show that

$$V = Ae^{-kt},$$

where k is a positive constant.

b Sketch a graph to show the relation between V and t.

Given further that $V = \frac{1}{2}A$ at $t = T$,

c show that $kT = \ln 2$. (Edexcel)

38 During a spell of freezing weather, the ice on a pond has thickness x mm at time t hours after the start of freezing. At 3.00 pm, after one hour of freezing weather, the ice is 2 mm thick and it is desired to predict when it will be 4 mm thick.

i In a simple model, the rate of increase of x is assumed to be constant. For this model, express x in terms of t and hence determine when the ice will be 4 mm thick.

ii In a more refined model, the rate of increase of x is taken to be proportional to $\frac{1}{x}$. Set up a differential equation for x, involving a constant of proportionality k.
Solve the differential equation and hence show that the thickness of ice is proportional to the square root of the time elapsed from the start of freezing.
Determine the time at which the second model predicts that the ice will be 4 mm thick. (OCR)

iii What assumption about the weather underlies both models?

39 i Express $\sin^2 x$ in terms of $\cos 2x$.

ii The region R is bounded by the part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ and the x-axis. Show that the volume of the solid formed when R is rotated completely about the x-axis is $\frac{1}{2}\pi^2$. (OCR)

40 i Find $\int xe^{-x} dx$.

ii A curve is such that at every point (x, y) on it the gradient, $\frac{dy}{dx}$, satisfies the differential equation

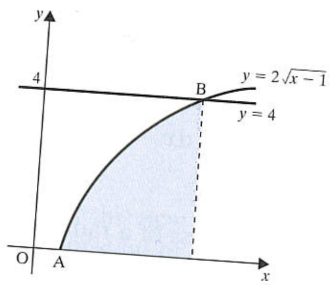
$$\frac{dy}{dx} = xe^{-x-y}.$$

Show that

$$e^y = \int xe^{-x} dx.$$

Given that the curve is defined for all values of x and that it passes through the origin, show that it also passes through the point $(-1, \ln 2)$. (AQA)

41 The graph of $y = 2\sqrt{x-1}$ is shown in the sketch, together with the line $y = 4$.



a Find the coordinates of the points A and B.

b Find the **exact** value of the shaded area shown.

c Calculate the **exact** volume of the solid formed by rotating the shaded area through 360° about the x-axis, leaving your answer in terms of π. (AQA)

a Use integration by parts to find $\int x \cos 2x \, dx$.

b By means of the substitution $u = 3x^2 + 1$, or otherwise, evaluate

$$\int_0^1 x\sqrt{3x^2 + 1} \, dx.$$

No credit will be given for a numerical approximation or for a numerical answer without supporting working. (AQA)

43 A curve has equation $y = \dfrac{3x+4}{(x-2)(2x+1)}$

a Express $\dfrac{3x+4}{(x-2)(2x+1)}$ in partial fractions.

b Show that $\dfrac{dy}{dx} = \dfrac{2}{(2x+1)^2} - \dfrac{2}{(x-2)^2}$ and hence, or otherwise, show that the curve has a turning point when $x = -3$.

Determine the value of x at the other stationary point of the curve.

c Find $\dfrac{d^2y}{dx^2}$ and hence determine the nature of the turning point when $x = -3$.

d Find $\displaystyle\int \frac{3x+4}{(x-2)(2x+1)}\,dx$.

Hence show that the area of the region bounded by the curve, the x-axis and the lines $x = 4$ and $x = 12$ is equal to $\ln 15$. (AQA)

44 The circle C has equation $x^2 + y^2 - 2x - 4y - 20 = 0$.

a Find the radius and the coordinates of the centre of C.

b Find the equation of the tangent to C at the point (4, 6), giving your answer in the form $ax + by + c = 0$. (AQA)

45 a Use the substitution $v = \sqrt{x^2+5}$ to evaluate: $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{x^2+5}}$

b Use integration by parts to find $\displaystyle\int_1^3 xe^{-2x}\,dx$. (OCR)

46 At time t hours, the rate of decay of the mass of a radioactive substance is proportional to the mass x kg at that time.

a Write down a differential equation satisfied by x.

b Given that $x = C$ when $t = 0$, show that $x = Ce^{-at}$ where a is a positive constant.

c Find the value of a if the mass halves every 2.5 hrs. (AQA)

47 The rate of destruction of a drug by the kidneys is proportional to the amount of the drug present in the body. The constant of proportionality is denoted by k. At time t the quantity of drug in the body is x. Write down a differential equation relating x and t, and show that the general solution is $x = Ae^{-kt}$, where A is an arbitrary constant.

Before $t = 0$ there is no drug in the body, but at $t = 0$ a quantity Q of the drug is administered. When $t = 1$ the amount of drug in the body is $Q\alpha$, where α is a constant such that $0 < \alpha < 1$. Show that $x = Q\alpha^t$.

Sketch the graph of x against t for $0 < t < 1$.

When $t = 1$ and again when $t = 2$ another dose Q is administered. Show that the amount of drug in the body immediately after $t = 2$ is $Q(1 + \alpha + \alpha^2)$. (OCR)

48 The parametric equations of a curve are

$$x = 3\cos\theta,\ y = 2\sin\theta \ \text{ for } \ 0 \leqslant \theta < 2\pi.$$

i By eliminating θ between these two equations, find the Cartesian equation of the curve.

ii Draw a sketch of the curve, giving the coordinates of the points where it cuts the axes. On your sketch show the pair of tangents which pass through the point (6, 2).

iii Use the parametric equations to calculate $\dfrac{dy}{dx}$ in terms of θ.

You are given that the equation of the tangent to the curve at $(3\cos\theta, 2\sin\theta)$ is

$$2x\cos\theta + 3y\sin\theta = 6.$$

iv Show that, for tangents to the curve which pass through the point (6, 2),

$$2\cos\theta + \sin\theta = 1.$$

v Solve the equation in **iv** to find the two values of θ (in radians correct to 2 decimal places) corresponding to the two tangents. (OCR)

49 A patch of oil pollution in the sea is approximately circular in shape. When first seen its radius was 100 m and its radius was increasing at a rate of 0.5 m per minute. At a time t minutes later, its radius is r metres. An expert believes that, if the patch is untreated, its radius will increase at a rate which is proportional to $1/r^2$.

i Write down a differential equation for this situation, using a constant of proportionality, k.

ii Using the initial conditions, find the value of k. Hence calculate the expert's prediction of the radius of the oil patch after 2 hours.

The expert thinks that if the oil patch is treated with chemicals then its radius will increase at a rate which is proportional to $\dfrac{1}{r^2(2+t)}$.

iii Write down a differential equation for this new situation and, using the same initial conditions as before, find the value of the new constant of proportionality.

iv Calculate the expert's prediction of the radius of the treated oil patch after 2 hours. (OCR)

50 i Use integration by parts to evaluate

$$\int 4x\cos 2x\,dx.$$

ii Use part **i**, together with a suitable expression for $\cos^2 x$, to show that

$$\int 8x\cos^2 x\,dx = 2x^2 + 2x\sin 2x + \cos 2x + c.$$

iii Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{8x\cos^2 x}{y}$$

which satisfies $y = \sqrt{3}$ when $x = 0$.

iv Show that any point (x, y) on the graph of this solution which satisfies $\sin 2x = 1$ also lies on one of the lines $y = 2x + 1$ or $y = -2x - 1$. (OCR)

EXAMINATION QUESTIONS G

1 Write down the expansions of the following expressions in ascending powers of x, as far as the term containing x^3. In each case state the values of x for which the expansion is valid.

i $(1-x)^{-1}$ **ii** $(1+2x)^{-2}$ **iii** $\dfrac{1}{(1-x)(1+2x)^2}$ (OCR)

2 Referred to an origin O, the position vectors of the points A and B are $-3\mathbf{i}+\mathbf{j}-7\mathbf{k}$ and $5\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ respectively. By using a scalar product, calculate, in degrees to one decimal place, the size of $\angle$AOB. (Edexcel)

3 The sequence given by the iteration formula

$$x_{n+1} = 100 + \ln x_n,$$

with $x_1 = 100$, converges to α. Find α correct to 2 decimal places, and write down an equation of which α is a root. (OCR)

4 Determine the coefficient of x^6 in the binomial expansion of

i $(1-x)^{15}$ **ii** $(1-x^2)^{15}$. (OCR)

5 The coefficient of x^3 in the expansion of $(1+2x)^n$ is $8n$, where n is a positive integer. Find the value of n. (OCR)

6 Expand $\left(x-\dfrac{1}{x}\right)^5$, simplifying the coefficients. (Edexcel)

7 Three points P, Q and R have position vectors, **p**, **q** and **r** respectively, where

$$\mathbf{p} = 7\mathbf{i}+10\mathbf{j},\quad \mathbf{q} = 3\mathbf{i}+12\mathbf{j},\quad \mathbf{r} = -\mathbf{i}+4\mathbf{j}.$$

i Write down the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$, and show that they are perpendicular.

ii Using a scalar product, or otherwise, find the angle PRQ.

iii Find the position vector of S, the midpoint of PR.

iv Show that $|\overrightarrow{QS}| = |\overrightarrow{RS}|$. Using your previous results, or otherwise, find the angle PSQ. (OCR)

8 a Obtain the first 4 non-zero terms of the binomial expansion in ascending powers of x of

$$(1-x^2)^{-\frac{1}{2}},\text{ given that } |x|<1,$$

b Show that, when $x = \frac{1}{3}$, $(1-x^2)^{-\frac{1}{2}} = \frac{3}{4}\sqrt{2}$.

c Substitute $x = \frac{1}{3}$ into your expansion and hence obtain an approximation to $\sqrt{2}$, giving your answer to 5 decimal places. (Edexcel)

9 Given that

$$(2-x)^{13} \equiv A + Bx + Cx^2 + \ldots,$$

find the values of the integers A, B and C. (Edexcel)

10 a By sketching the curves with equations $y = 4 - x^2$ and $y = e^x$, show that the equation $x^2 + e^x - 4 = 0$ has one negative root and one positive root.

b Use the iteration formula $x_{n+1} = -(4-e^{x_n})^{\frac{1}{2}}$ with $x_0 = -2$ to find in turn x_1, x_2, x_3 and x_4 and hence write down an approximation to the negative root of the equation, giving your answer to 4 decimal places.

An attempt to evaluate the positive root of the equation is made using the iteration formula $x_{n+1} = (4-e^{x_n})^{\frac{1}{2}}$ with $x_0 = 1.3$.

c Describe the result of such an attempt. (Edexcel)

11 a Rearrange the cubic equation $x^3 - 6x - 2 = 0$ into the form

$$x = \pm\sqrt{a+\frac{b}{x}}.$$

State the values of the constants a and b.

b Use the iterative formula $x_{n+1} = \sqrt{a+\dfrac{b}{x_n}}$ with $x_0 = 2$ and your values of a and b to find the approximate positive solution x_4 of the equation to an appropriate degree of accuracy. Show all your intermediate answers. (Edexcel)

12 a Expand $(1-2x)^{10}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient in the expansion.

b Use your expansion to find an approximation to $(0.98)^{10}$, stating clearly the substitution which you have used for x. (Edexcel)

13 a Expand $(3+2x)^4$ in ascending powers of x, giving each coefficient as an integer.

b Hence, or otherwise, write down the expansion of $(3-2x)^4$ in ascending powers of x.

c Hence by choosing a suitable value for x show that $(3+2\sqrt{2})^4 + (3-2\sqrt{2})^4$ is an integer and state its value. (Edexcel)

14 a Use the iteration $x_{n+1} = (3x_n+3)^{\frac{1}{3}}$, with $x_0 = 2$, to find, to 3 significant figures, x_4. The only real root of the equation $x^3 - 3x - 3 = 0$ is α. It is given that, to 3 significant figures, $\alpha = x_4$.

b Use the substitution $y = 3^x$ to express

$$27^x - 3^{x+1} - 3 = 0$$

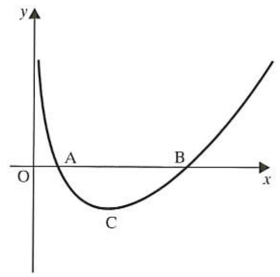
as a cubic equation.

c Hence, or otherwise, find an approximate solution to the equation

$$27^x - 3^{x+1} - 3 = 0,$$

giving your answer to 2 significant figures. (Edexcel)

15



The figure shows the curve with equation $y = x - \ln x - 2$. The curve crosses the x-axis at A and B, and C is the minimum point on the curve.

a Find the coordinates of point C.

b Given the iteration $u_{n+1} = \ln u_n + 2$, and $u_0 = 3$, find, to three significant figures, u_4. (You are advised to show all stages of your working.)

c Given the iteration $v_{n+1} = e^{v_n - 2}$, and $v_0 = 0.5$, find, to three significant figures, v_4.

d Given that $u_n \to u$ as $n \to \infty$, show that u satisfies the equation $x - \ln x - 2 = 0$.

e Given that $v_n \to v$ as $n \to \infty$, show that v satisfies the equation $x - \ln x - 2 = 0$.

f Hence estimate the coordinates of points A and B, giving your answers to 3 significant figures.

(Edexcel)

16 i Write down the expansion of $(2 - x)^4$.

ii Find the first four terms in the expansion of $(1 + 2x)^{-3}$ in ascending powers of x. For what range of values of x is this expansion valid?

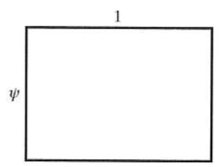
iii When the expansion is valid,

$$\frac{(2-x)^4}{(1+2x)^3} = 16 + ax + bx^2 + \ldots$$

Find the values of a and b.

(OCR)

17 A golden rectangle has one side of length 1 unit and a shorter side of length ψ units, where ψ is called the golden section.



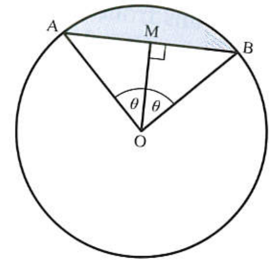
ψ can be found using the iterative formula

$$x_{n+1} = \sqrt[3]{x_n(1 - x_n)}.$$

Choosing a suitable value for x_1 and showing intermediate values, use this iterative formula to obtain the value of ψ to 2 decimal places.

(AQA)

18 The figure shows a circle of radius 1 and centre O.



Given that the shaded area is $\frac{1}{6}$ of the area of the complete circle, show that

$$\theta - \frac{1}{2}\sin 2\theta = \frac{\pi}{6}.$$

Show that θ lies between 0.9 and 1.1 radians.

An iterative procedure that can be used to find θ is based on the sequence

$$\theta_{n+1} = \frac{1}{2}\sin 2\theta_n + \frac{\pi}{6}.$$

Taking $\theta_0 = 1$, find θ correct to 2 decimal places.

(WJEC)

19 Three points have coordinates A(9, 2, −4), B(3, 1, −4) and C(2, 7, 6).

a i Write down the vector $\overrightarrow{AB}$.

ii Write down the vector $\overrightarrow{BC}$.

iii Show that AB is perpendicular to BC.

b i Write down a vector equation for the line L which is parallel to BC and which passes through the point (−4, 6, 6).

ii Show that L intersects the line AB.

(AQA)

20 a Illustrate, by sketching graphs of

$$y = \ln(5x) \quad \text{and} \quad y = \frac{10}{x}, \; x > 0$$

on the same diagram, that the equation

$$x \ln(5x) - 10 = 0$$

has just one real root.

b Show, by using Newton's method, that when a is an approximation to this root then a better approximation is usually given by

$$\frac{a + 10}{1 + \ln(5a)}.$$

c Use this result twice, starting with $a = 3$, to find a further approximation giving three decimal places in your answer.

(AQA)

21 a Express $\dfrac{1 - x - x^2}{(1-2x)(1-x)^2}$ as the sum of three partial fractions.

b Hence, or otherwise, expand this expression in ascending powers of x up to and including the term in x^3.

c State the range of values of x for which the full expansion is valid.

(AQA)

22 A pipe runs from the point A(2, 1, 5) to the point B(6, 0, 6). At B the pipe bends and then runs to the point C(7, 0, 3).

a Find a vector equation of the line AB.

b Find the angle between the vectors

$$\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}.$$

c Find the angle ABC formed by the pipe. (AQA)

23 Given that $|x| < 1$, expand $\sqrt{1+x}$ as a series of ascending powers of x, up to and including the term in x^2.

Show that, if x is small, then

$$(2-x)\sqrt{1+x} \approx a + bx^2,$$

where the values of a and b are to be stated. (OCR)

24 A curve C has equation $y = \frac{\sin x}{x}$, where $x > 0$.

i Find $\frac{dy}{dx}$, and hence show that the x-coordinate of any stationary point of C satisfies the equation $x = \tan x$.

ii Use the iteration $x_{n+1} = \pi + \tan^{-1} x_n$ to find, correct to 2 decimal places, the root of the equation $x = \pi + \tan^{-1} x$ which lies between 4 and 5.

iii Show that every root of $x = \pi + \tan^{-1} x$ is also a root of $x = \tan x$. (OCR)

25 Find, in their simplest form, the first three terms in the expansion of

$$(1+3t)^{\frac{2}{3}}$$

in ascending powers of t, where $|t| < \frac{1}{3}$. (AQA)

26 Referred to a fixed origin O, the points A and B have position vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 9\mathbf{k}$ respectively.

a Show that OA is perpendicular to AB.

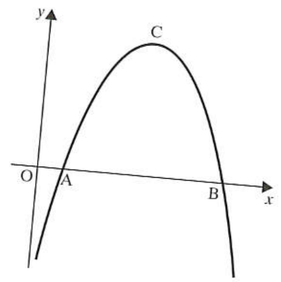
b Find, in vector form, an equation of the line L_1 through A and B.

The line L_2 has equation $\mathbf{r} = (8\mathbf{i} + \mathbf{j} - 6\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$, where λ is a scalar parameter.

c Show that the lines L_1 and L_2 intersect and find the position vector of their point of intersection. (Edexcel)

27 Write down and simplify the binomial expansion of $(1-2x)^{\frac{1}{3}}$ up to and including the x^3 term. By putting $x = \frac{1}{10}$, use your expansion to find an approximation to the cube root of 10, giving your answer as the ratio of two integers. (AQA)

28



The function f is defined by

$$f(x) = 18 \ln x - x^2, \quad x > 0$$

The diagram shows a sketch of the curve with equation $y = f(x)$.

i The point C is the maximum point of the curve. Find the x-coordinate of C.

ii The curve crosses the x-axis at the points A and B. Show by calculation that the x-coordinate of A lies between 1.0 and 1.1.

iii Using $x = 1.0$ as a first approximation to the x-coordinate of A, apply the Newton–Raphson method twice to $f(x) = 0$ to obtain a third approximation, giving your answer correct to three decimal places.

iv The normal at the point $P(2, 18\ln 2 - 4)$ meets the x-axis at the point N. Find the area of the triangle OPN, where O is the origin, giving your answer correct to three significant figures. (OCR)

29 On a single diagram, sketch the graphs of $y = \ln(10x)$ and $y = \frac{6}{x}$, and explain how you can deduce that the equation $\ln(10x) = \frac{6}{x}$ has exactly one real root.

Given that the root is close to 2, use the iteration

$$x_{n+1} = \frac{6}{\ln(10x_n)}$$

to evaluate the root correct to three decimal places.

The same equation may be written in the form $x\ln(10x) - 6 = 0$. Taking $f(x)$ to be $x\ln(10x) - 6$, find $f'(x)$, and show that the Newton–Raphson iteration for the root of $f(x) = 0$ may be simplified to the form

$$x_{n+1} = \frac{x_n + 6}{1 + \ln(10x_n)}$$

(OCR)

30 Show that the equation $x^3 - x^2 - 2 = 0$ has a root a which lies between 1 and 2.

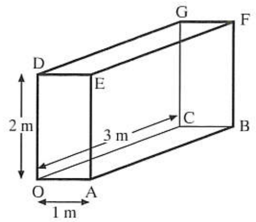
a Using 1.5 as a first approximation for a, use the Newton–Raphson method once to obtain a second approximation for a, giving your answer to 3 decimal places.

b Show that the equation $x^3 - x^2 - 2 = 0$ can be arranged in the form $x = \sqrt[3]{f(x)}$ where $f(x)$ is a quadratic function.

Use an iteration of the form $x_{n+1} = g(x_n)$ based on this rearrangement and, with $x_1 = 1.5$, to find x_2 and x_3, giving your answers to 3 decimal places. (AQA)

Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$. (OCR)

32

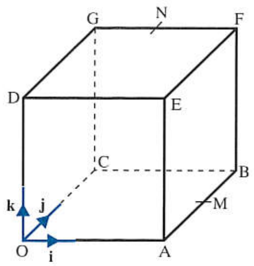


The figure shows a cuboid in which OA = 1 m, OC = 3 m and OD = 2 m. Taking O as origin and unit vectors **i**, **j**, **k** in the directions OA, OC, OD respectively, express in terms of **i**, **j**, **k** the vectors

i OF and **ii** AG

By considering an appropriate scalar product, find the acute angle between the diagonals OF and AG. (WJEC)

33



In the diagram OABCDEFG is a cube in which the length of each edge is 2 units.
Unit vectors **i**, **j**, **k** are parallel to $\overrightarrow{OA}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ respectively. The midpoints of AB and FG are M and N respectively.

i Express each of the vectors $\overrightarrow{ON}$ and $\overrightarrow{MG}$ in terms of **i**, **j** and **k**.

ii Show that the acute angle between the direction of $\overrightarrow{ON}$ and $\overrightarrow{MG}$ is 63.6°, correct to the nearest 0.1°. (OCR)

34 Given that $y = \ln(4+3x)$, find $\dfrac{dy}{dx}$ and show that $\dfrac{d^2y}{dx^2} = -\frac{9}{16}$ when $x = 0$.

Hence, or otherwise, obtain the Maclaurin series for $\ln(4+3x)$, up to and including the term in x^2. (OCR)

35 Given that $y = \cos\left(\frac{1}{3}\pi + 2x\right)$, find $\dfrac{d^2y}{dx^2}$.

Hence obtain the Maclaurin series for $\cos\left(\frac{1}{3}\pi + 2x\right)$, up to and including the term in x^2. (OCR)

36 Obtain the first three terms in the Maclaurin series for $\ln(3+x)$. (OCR)

37 A curve is defined by the parametric equations $x = \cos^{-1}(t)$, $y = -\ln t$.

a Find $\dfrac{dt}{dx}$. Hence show that $\dfrac{dy}{dx} = \tan x$.

b Use the Maclaurin Series to find an approximation to the curve near the origin, of the form

$$y = a + bx + cx^2.$$

(AQA)

38 a Given that $|x| < \frac{1}{3}$, write down the expansion of $(1-3x)^{-2}$ in ascending powers of x up to and including the term x^3.

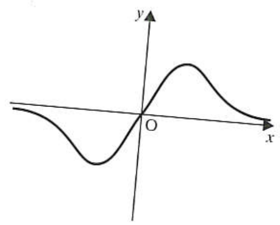
b Write down the series expansions of

i e^{4x} **ii** $\sin 2x$

in ascending powers of x up to and including the term in x^3 simplifying the coefficients as much as possible.

c Given that x is small enough for x^4 and higher powers to be ignored, show that $(1-3x)^{-2} - e^{4x} - \sin 2x = ax^2 + bx^3$ and state the values of the constants a and b. (AQA)

39 i Show that $\dfrac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$.



The sketch above shows the curve with equation $y = xe^{-x^2}$.

ii Differentiate xe^{-x^2} and find the coordinates of the two stationary points on the curve.

iii Find the area of the region between the curve and the x-axis for $0 \leqslant x \leqslant 0.4$.

iv Give the first two terms of the series expansion for e^{-x^2} and hence find an approximation for $y = xe^{-x^2}$ when x is small.

v Use your answer to part **iv** to find an approximation for the area which you calculated in part **iii**. (OCR)

40 Referred to a three-dimensional Cartesian coordinate system with origin O, the points A, B and C have coordinates (4, 0, 0), (0, 4, 0) and (0, 0, 5) respectively.
Calculate the acute angle between the planes OAB and CAB, giving your answer in degrees to 1 decimal place. (Edexcel)

41 A plane Π has equation $ax + by + z = d$.

i Write down, in terms of a and b, a vector which is perpendicular to Π.

Points A(2, −1, 2), B(4, −4, 2), C(5, −6, 3) lie on Π.

ii Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

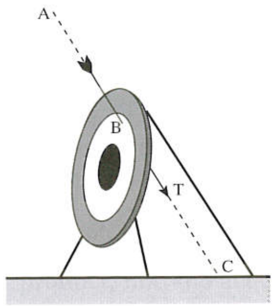
iii Use scalar products to obtain two equations for a and b.

iv Find the equation of the plane Π.

v Find the angle which the plane Π makes with the plane $x = 0$.

vi Point D is the midpoint of AC. Point E is on the line between D and B such that DE : EB = 1 : 2. Find the coordinates of E. (OCR)

42 The figure shows an arrow embedded in a target. The line of the arrow passes through the point A(2, 3, 5) and has direction vector $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$. The arrow intersects the target at the point B. The plane of the target has equation $x+2y-3z=4$. The units are metres.



i Write down the vector equation of the line of the arrow in the form

$$\mathbf{r}=\mathbf{p}+\lambda\mathbf{q}.$$

ii Find the value of λ which corresponds to B. Hence write down the coordinates of B.

iii The point C is where the line of the arrow meets the ground, which is the plane $z=0$. Find the coordinates of C.

iv The tip, T, of the arrow is one-third of the way from B to C. Find the coordinates of T and the length of BT.

v Write down a normal vector to the plane of the target. Find the acute angle between the arrow and this normal.

(OCR)

43 The position vectors of three points A, B C on a plane ski-slope are

$$\mathbf{a}=4\mathbf{i}+2\mathbf{j}-\mathbf{k},\quad \mathbf{b}=-2\mathbf{i}+26\mathbf{j}+11\mathbf{k},\quad \mathbf{c}=16\mathbf{i}+17\mathbf{j}+2\mathbf{k}$$

where the units are metres.

i Show that the vector $2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$ is perpendicular to $\overrightarrow{AB}$ and also perpendicular to $\overrightarrow{AC}$. Hence find the equation of the plane of the ski-slope.

The track for an overhead railway lies along the straight line DEF, where D and E have position vectors $\mathbf{d}=130\mathbf{i}-40\mathbf{j}+20\mathbf{k}$ and $\mathbf{e}=90\mathbf{i}-20\mathbf{j}+15\mathbf{k}$, and F is a point on the ski-slope.

ii Find the equation of the straight line DE.

iii Find the position vector of the point F.

iv Show that $\overrightarrow{FD}=15(8\mathbf{i}-4\mathbf{j}+\mathbf{k})$ and hence find the length of the track.

(OCR)

44 In a crystal structure one of the planes of the crystal has equation

$$2x-3y+z=5.$$

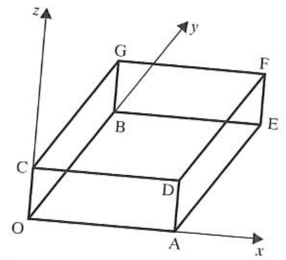
An X-ray beam passes through the crystal along the line having equation

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}+\lambda\begin{bmatrix}3\\1\\1\end{bmatrix}.$$

Find the angle between the X-ray beam and the given plane.

(AQA)

45



A rectangular-faced box OADCBEFG, has sides of length 5 cm, 4 cm, 3 cm. A rectangular Cartesian coordinate system has its origin O, as shown above. The following are the coordinates of A, B and C:

$$A:(5,0,0);\quad B:(0,4,0);\quad C:(0,0,3).$$

a Write down the coordinates of the point G.

b Calculate the length of the line AG.

c Find, in an appropriate form, the equations of the line AG.

d The plane with equation $24x-16y+20z=244$ meets the line AG in a point P. Prove that the coordinates of P are $(10,-4,-3)$.

(OCR)

46 The Cartesian coordinates of three points in three-dimensional space are

$$A=(1,0,2),\quad B=(3,1,0)\quad\text{and}\quad C=(2,-1,1).$$

a Find, in the form $ax+by+cz=d$, the equation of the plane containing the points A, B and C.

b Find, in the form $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, the equations of the line that passes through A and B.

(OCR)