Lesson 1: Natural numbers

* Contents:

  1. Number systems. Positional notation.
  2. Basic arithmetic. Algorithms and properties.
  3. Algebraic language and abstract reasoning.
  4. Divisibility. Prime numbers. Greatest common divisor. Least common multiple.

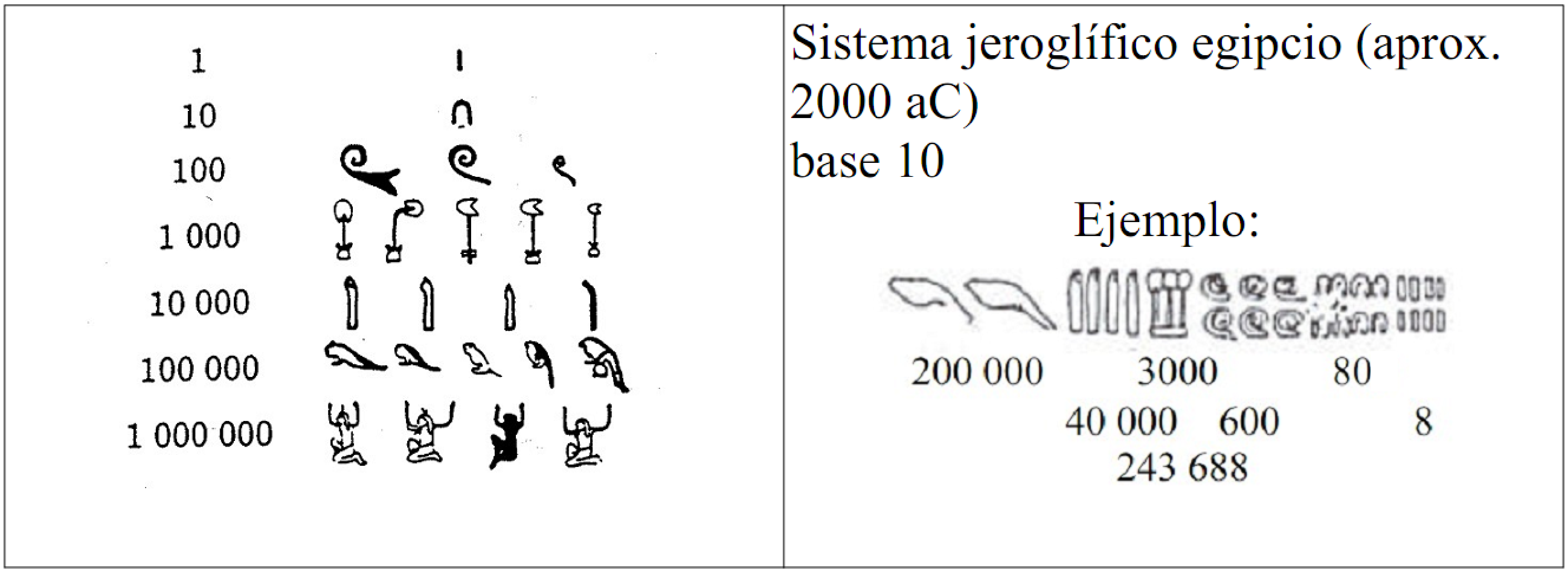
Natural numbers

* $\mathbb{N} = \{1, 2, 3, 4, 5 \ldots\}$
* Origin: need to “count”.
* Problem: (word and number) representation of “big” numbers’.

Types of number systems

1. Aditive systems

* Number is obtained adding up the value of the symbols.



From http://www.ugr.es/ jgodino/edumat-maestros/welcome.htm

* Greek number system: I $= 1$, $\Pi = 5$, $\Delta = 10$, H $= 100$, X $= 1000$ and M $= 10000$ (roman numerals come from it).

Types of number systems

2. Aditive-multiplicative systems

* Instead of repeating a symbol several times, an extra symbol is added to indicate that.

  An example: the chinese system

一	二	三	四	五	六	七	八	九	十	百	千	萬
1	2	3	4	5	6	7	8	9	10	100	1 000	10 000

De esta manera se evitan repeticiones fastidiosas pues los números que preceden a las potencias de la base indican cuántas veces deben repetirse éstas. Por ejemplo, el número 79564 se escribiría:

七 萬 九 千 五 百 六 十 四

qī wàn jiǔ qiān wǔ bǎi liù shí sì

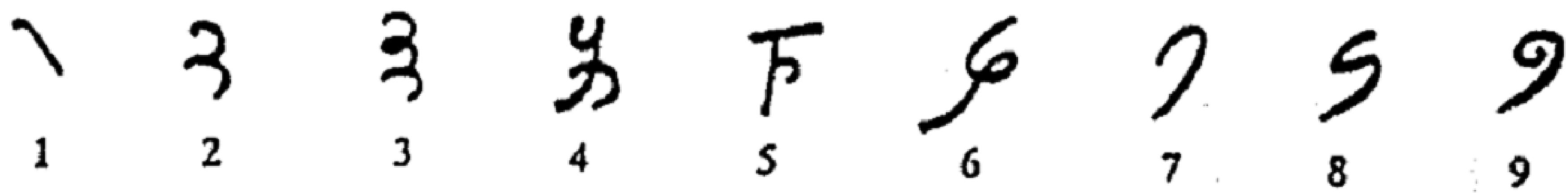
$7 \times 10000 + 9 \times 1000 + 5 \times 100 + 6 \times 10 + 4$

79 564

Types of number systems

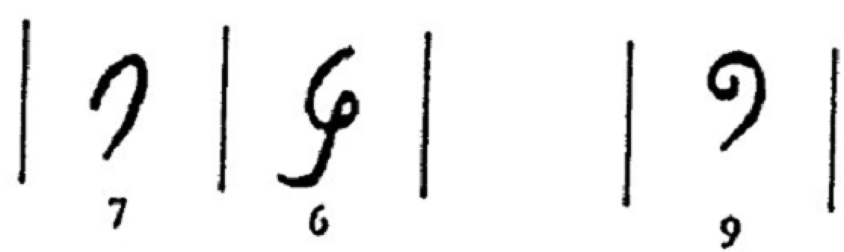
3. Multiplicative systems (like ours)

* Origin: Hindu system. Symbols were



(and some additional ones for powers of 10).

* Around 5h-8th centuries, symbols for powers of 10 are substituted by bars:



Number systems

- The symbol 0 (zero).

  The name comes from sanscrit word shunya (empty). Translated to arabic as sifr. (Origin of the Spanish word palabra cifra).

  The system arrives at Europe via muslims (hindu-arabic system). Al-Jwarizmi wrote the book "The Book of Addition and Subtraction According to the Hindu Calculation" around 825.

- With the introduction of the new number system, arithmetic develops very fast.

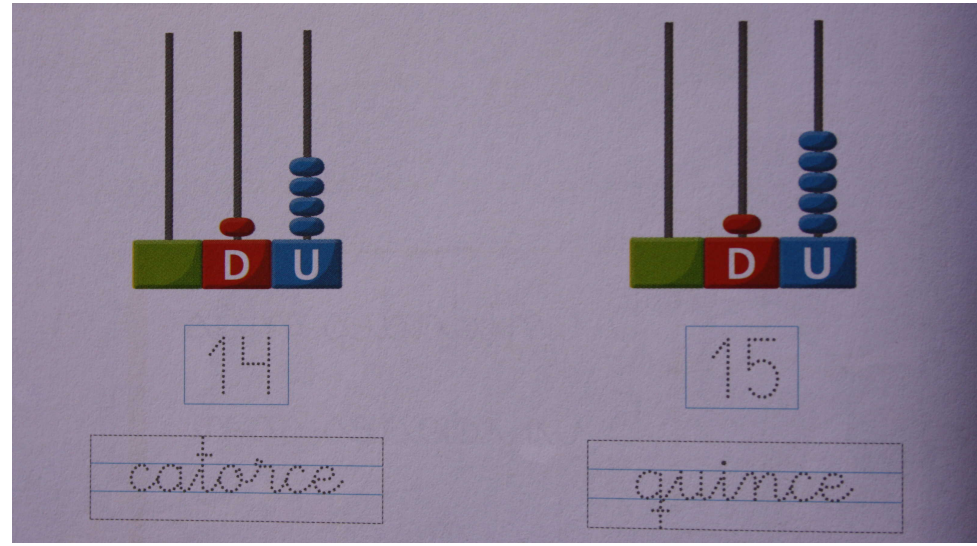
Two digit numbers in 1st Grade

* Traditional approach (in Spain):
  - Lesson 0: review of numbers from 0 to 9.
  - Lesson 1: numbers from 10 to 19.
  - Lesson 2: numbers from 20 to 29.
  - . . .

* Drawback: Lack of number sense.

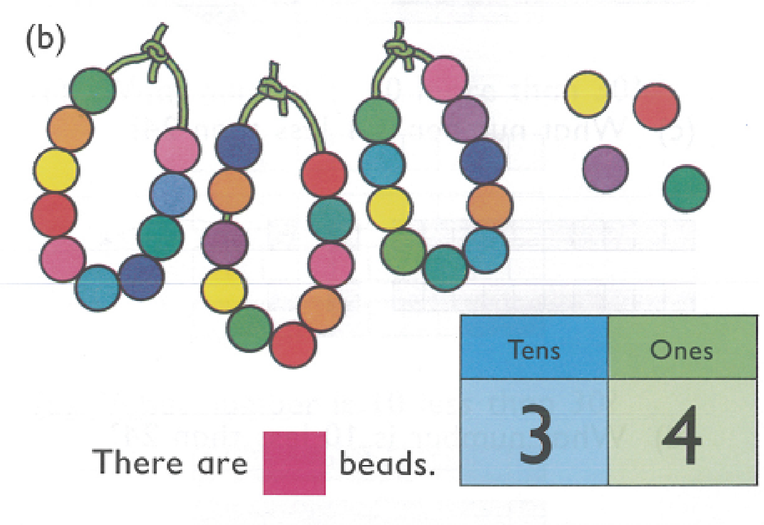
Tens and units in Spanish textbooks

* Usual approach, as in the figure:



* It is better, at least for some time, represent tens explicitely, as in the figure:

  (Example extracted from book 1B of Singapore).

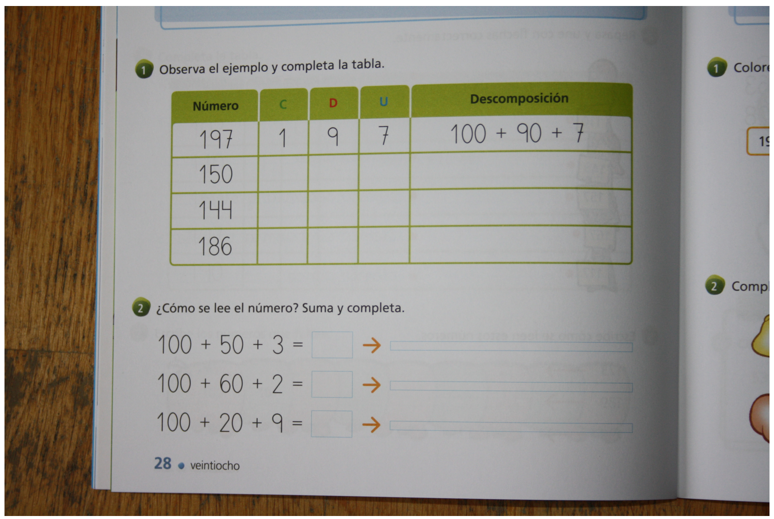


A comparison

* Let us compare these two examples:

From Singapore K-3

Spanish textbook, K-2



1 Observa el ejemplo y completa la tabla.

Número	C	D	U	Descomposición
197	1	9	7	100 + 90 + 7
150				
144				
186				

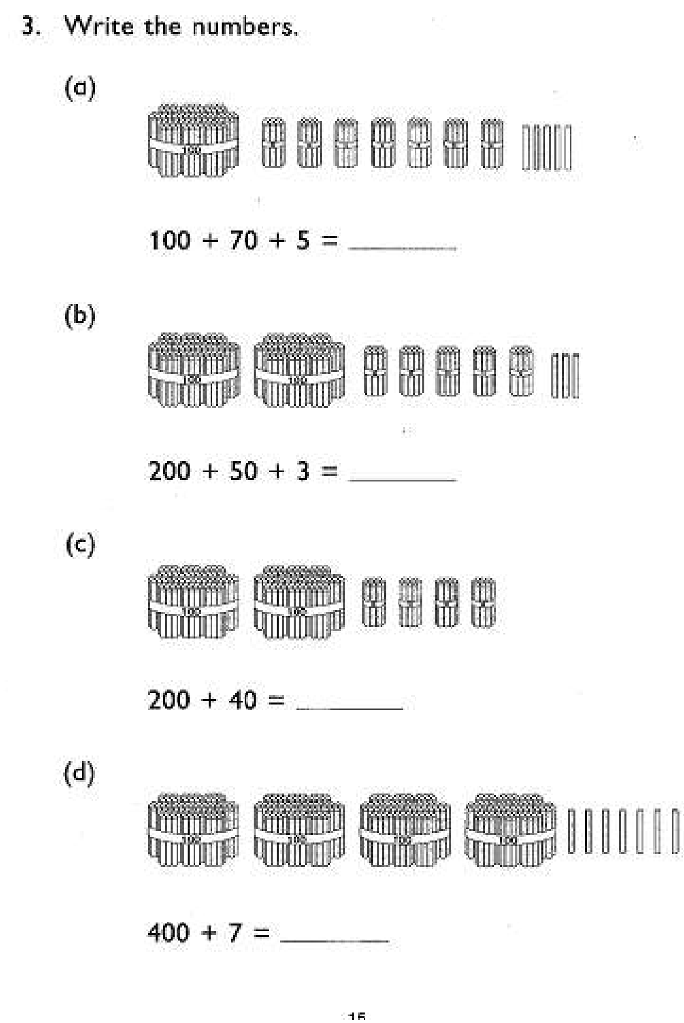
2 ¿Cómo se lee el número? Suma y completa.

100 + 50 + 3 = →

100 + 60 + 2 = →

100 + 20 + 9 = →

28 • veintiocho



3. Write the numbers.

(a) 100 + 70 + 5 = ______

(b) 200 + 50 + 3 = ______

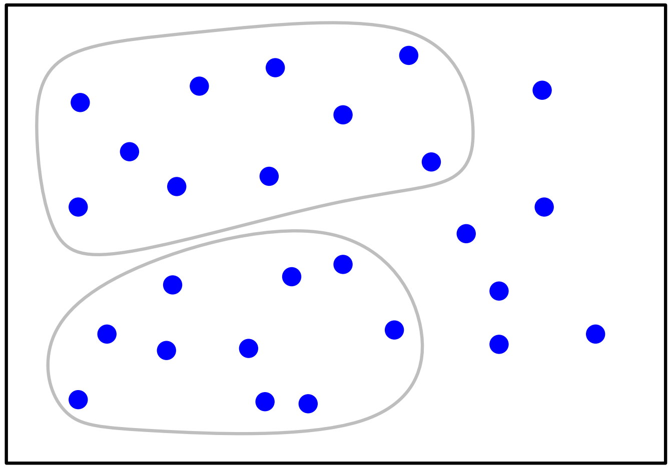
(c) 200 + 40 = ______

(d) 400 + 7 = ______

15

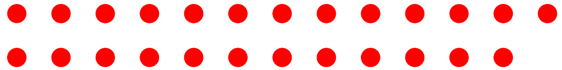
Introducing 2-digit numbers

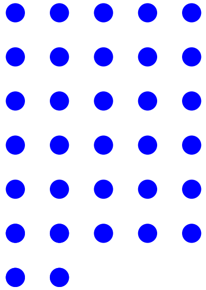
- Alternative approach: we count "making groups of ten".



There are two "groups of ten" and 6 (two tens and 6).

- Furthermore, examples for counting can be chosen in order to develop the number sense.





Introducing 2-digit numbers

- Once enough practice on "counting by tens" has been made, next step would be:
  - 3 tens (groups of ten) and 5 is written 35.
  - the group of ten is called "decena".
  - introduce the symbol 0.

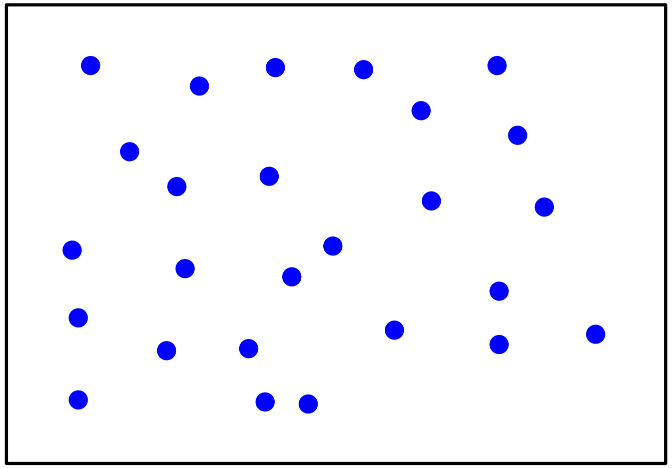
- After these steps, a student is ready to answer a question like: how many are $32 + 20$?

- This topic (and its relationship with the introduction of addition and substraction algorithms) will be revisited in more detail next year, in "Didactics of Mathematics".

Base b

* Why do we count in base ten (making "groups de ten")?

* If human beings had 8 fingers, how the number of dots in the figure would be represented?



* Because $26 = 3 \times 8 + 2$, in base 8 the number 26 is written as $32_{(8}$.

* Exercise: How will you write number 26 in base 5?

Base b

* Expression of a number in base b: given a natural numberl $b > 1$ (the base), every number n can be written in a unique way as follows

$$n = a_k \cdot b^k + a_{k-1} \cdot b^{k-1} + \cdots + a_1 \cdot b + a_0.$$

where digits a_i are natural numbers from 0 to $b - 1$.

The expression of n in base b is $a_k a_{k-1} \cdots a_1 a_{0(b}$.

* Why base b is interesting?
  - Didactics.
  - Conection with math applications: Computer Science.

Base b: exercises

* Exercices:
  1. Write the first five natural numbers in base 2.
  2. Write (in base 4) the 10 numbers following $223_{(4}$.
  3. How to convert from base 10 to base b, and conversely?
     (a) Write $354_{(7}$ in base 10.
     (b) Write 92 in base 3.

Oral number system

* Write in letters
  * $87\,065\,006$.
  * $72\,080\,023\,002\,305\,006$.

Write with digits twenty-three thousand forty-three billion, two hundred and four thousand million, twenty thousand and four.

More information here:

http://goo.gl/XJiZo (Wikipedia) (Spanish system)

* Ordinal numbers

Write with letters 37^{o}, 76^{o}, 85^{o}, 94^{o}, 101^{o}.

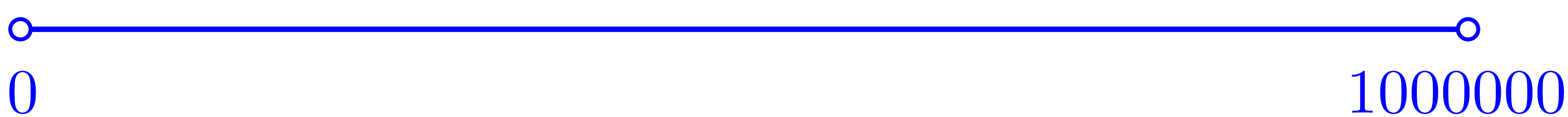
The number line

- An excellent tool for developing the number sense (at every level).

- For instance: at the end of 1st or 2nd year:

  Find the approximate place for numbers 87, 6, 25, 48.

- At the end of elementary school, same thing for 870100, 6005, 250037, 48025.



Addition - Concept and algorithms

* Too many times, the study of addition is reduced to the study of the traditional algorithm.

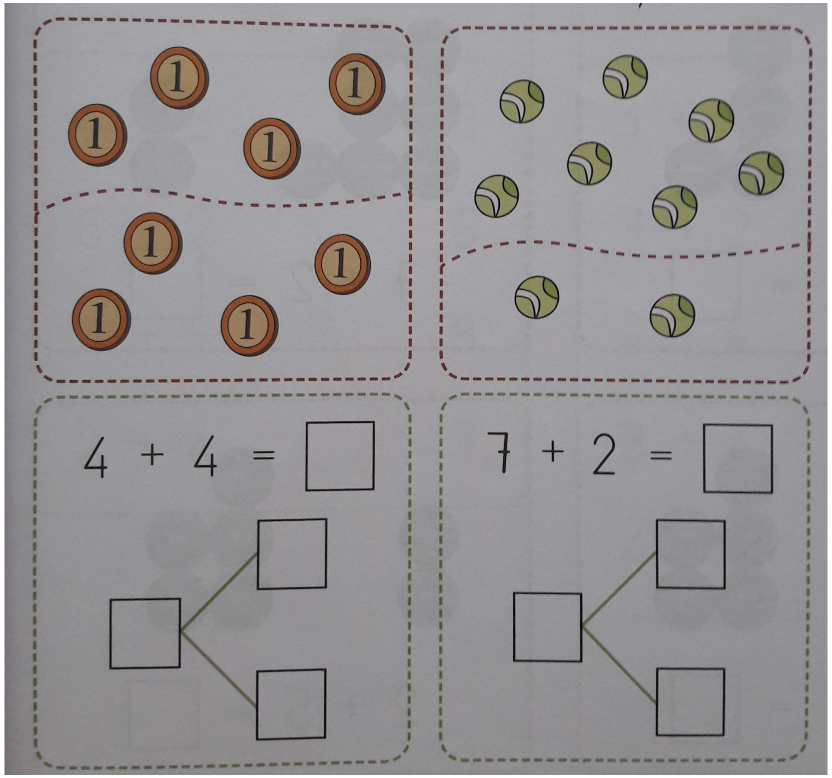
$$\begin{array}{r} {}^{1} \\ 37 \\ +\;25 \\ \hline 62 \end{array} \qquad \begin{array}{r} {}^{1\,1} \\ 813674 \\ +\;452895 \\ \hline 1266569 \end{array}$$

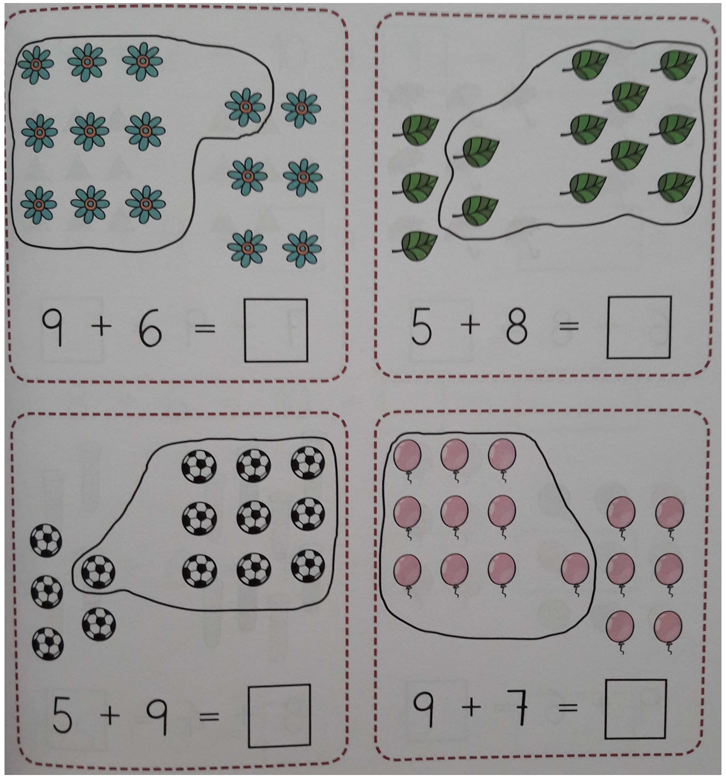
* Before going on, it would be worth to stop and think over the role of traditional arithmetic in basic mathematics of this century.

Mas ideas, menos cuentas: La aritmética en primaria

A short comment on didactics

* Before studing the traditional algorithm, it is important to develop the number sense with 1-digit numbers.



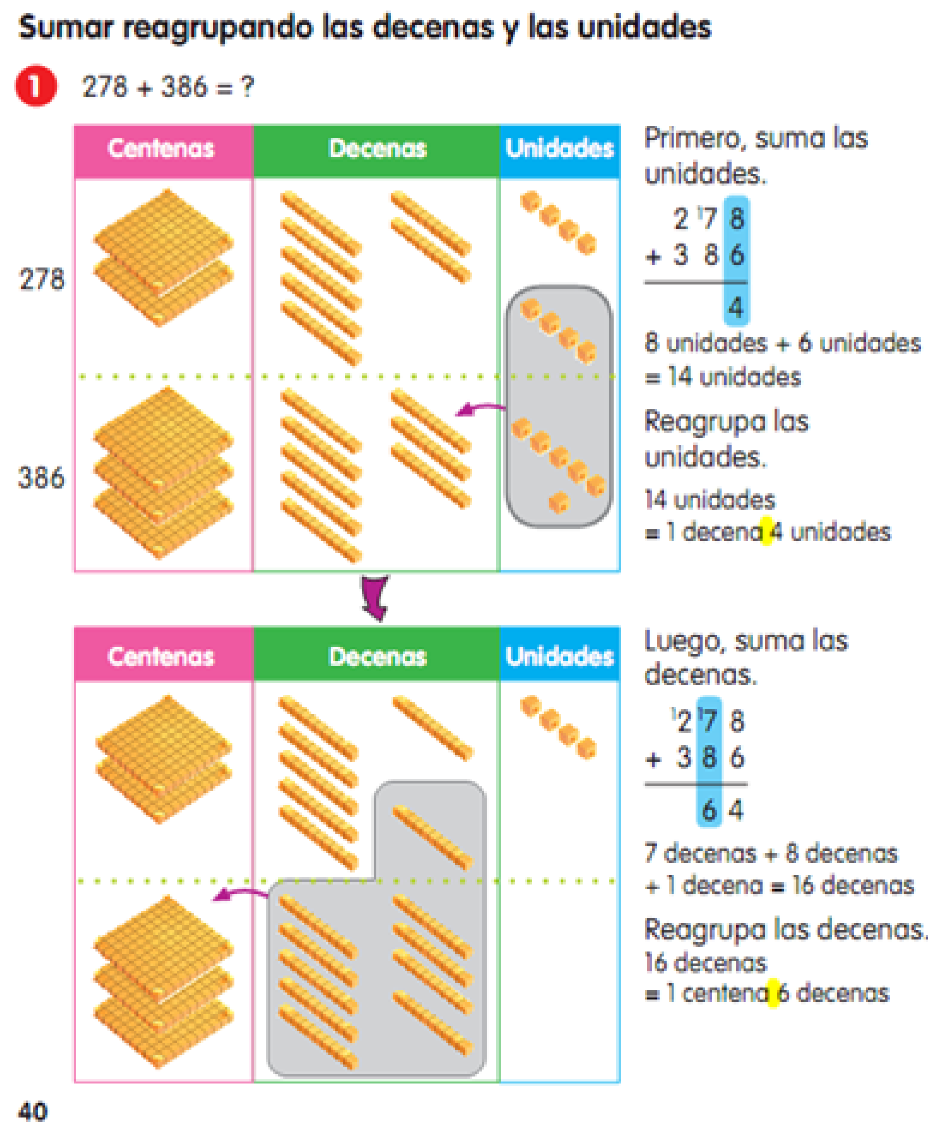


* The main mistake in Spain: To introduce too soon, and too fast, column (traditional) algorithms.

Addition algorithms

$$
\begin{array}{cccccccc}
 & 1 & 1 & 1 & 1 & & 1 & \\
 & 2 & 8 & 3 & 7 & 4 & 4 & 6 \\
+ & & 9 & 8 & 3 & 7 & 4 & 5 \\
\hline
 & 3 & 8 & 2 & 1 & 1 & 9 & 1
\end{array}
$$

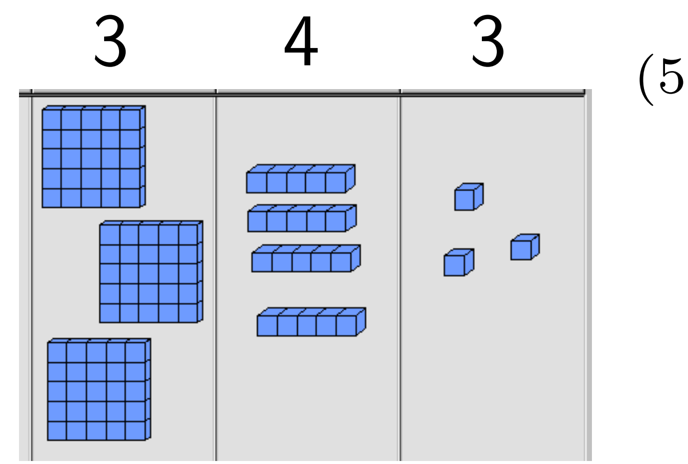
Obviously, the way an algorithm is presented is also very important.

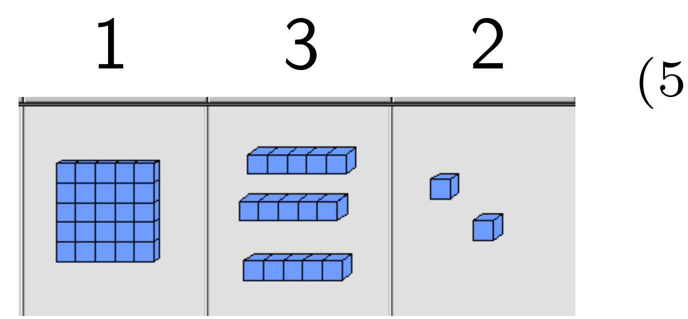


Addition in base b

- A good way of cheking if we really understand carrying (llevadas o reagrupamientos) is to compute this addition in base 5.

$$\begin{array}{cccc} & 3 & 4 & 3_{(5} \\ + & 1 & 3 & 2_{(5} \\ \hline \end{array}$$





Images obtained from National library of virtual manipulatives:
http://nlvm.usu.edu/en/nav/vlibrary.html

Numbers and operations → Base blocks

(Java is needed)

Exercises

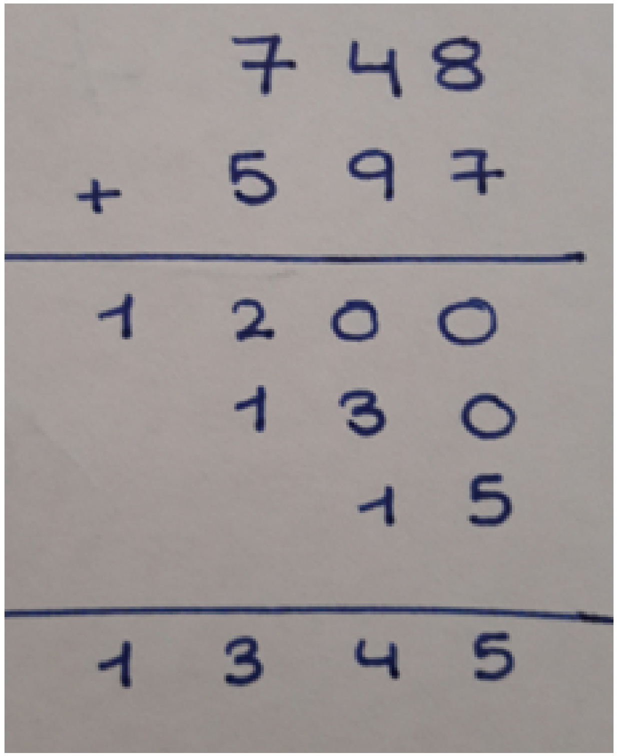
* Compute this base 6 addition, double-checking that you understand the carrying that you do in the process.

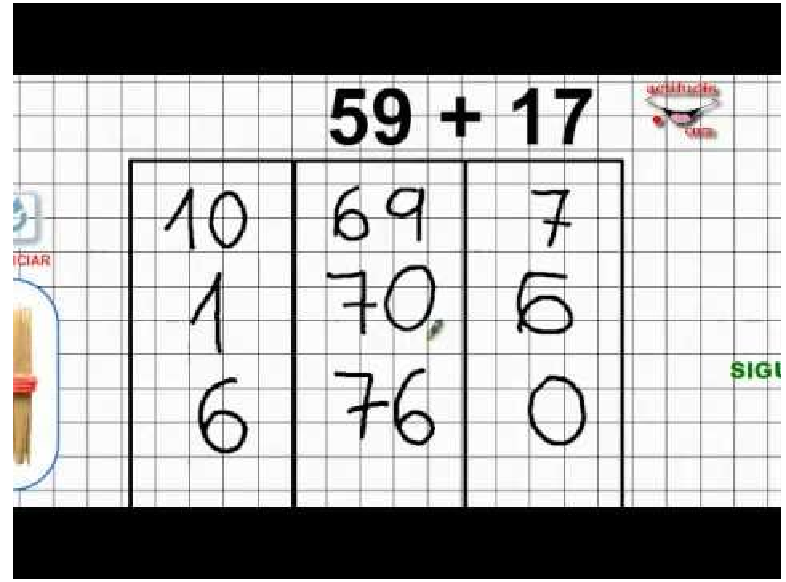
$$\begin{array}{rccccc} & 5 & 4 & 3 & 2 & 3_{(6} \\ + & 2 & 1 & 5 & 1 & 4_{(6} \\ \hline \end{array}$$

* Fill in the squares in the following base 8 addition.

$$\begin{array}{cccccc} & 5 & \square & 2 & 6 & \square_{(8} \\ + & \square & 2 & \square & 3 & 4_{(8} \\ \hline 1 & 3 & 0 & 4 & \square & 1_{(8} \end{array}$$

Some alternatives





ABN algorithm

* Compute the following additions using these algorithms and analyze their advantages and drawbacks.

a) $89 + 75$ b) $528 + 849$

Basic arithmetic: addition and substraction

* Addition is an internal operation in $\mathbb{N}$.
* Properties: conmutative, associative.
* Unce addition has been defined, substraction is easy:

  We say that $a - b = c$ if $b + c = a$.

  Comment: understanding substraction in this way right from the beginning has important consequences. For instance, makes clear the symmetric role of b and c in the expression $a - b = c$. More en didactics.
* Substraction is not an internal operation in $\mathbb{N}$.
* Using substraction, the order in $\mathbb{N}$ can be defined:

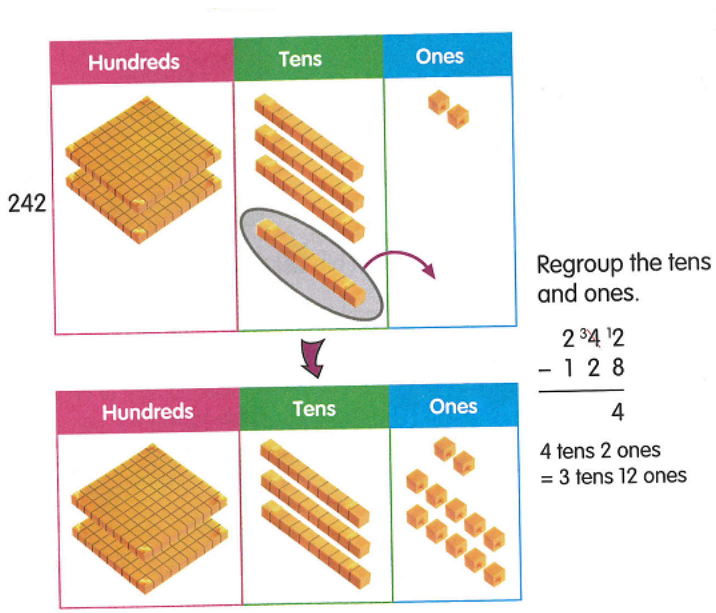
  we say that $a < b$ if $b - a \in \mathbb{N}$.

Substraction algorithms

* (Our) Standard algorithm: how does it work?

$$\begin{array}{r} 2\;4\;2 \\ -\;1\;2\;8 \\ \hline \end{array}$$

* An alternative (Asia, English speaking countries, already common in Spain)



$$\begin{array}{r} 2\;4\;2 \\ -\;1\;2\;8 \\ \hline \end{array}$$

Substraction in base b

- Again, computing in base b is a good tool to think about the algorithms.

$$\begin{array}{rccc} & 3 & 1 & 2_{(5} \\ - & 1 & 3 & 4_{(5} \\ \hline \end{array}$$

- A small disadvantage of the "international": zeros in "minuendo".

$$\begin{array}{rccc} & 3 & 0 & 1_{(4} \\ - & 1 & 2 & 3_{(4} \\ \hline \end{array}$$

- Compute this substraction in base 6, using both algorithms, and paying special attention to understand the carrying.

$$\begin{array}{rccccc} & 5 & 0 & 2 & 5 & 3_{(6} \\ - & 2 & 3 & 5 & 1 & 4_{(6} \\ \hline \end{array}$$

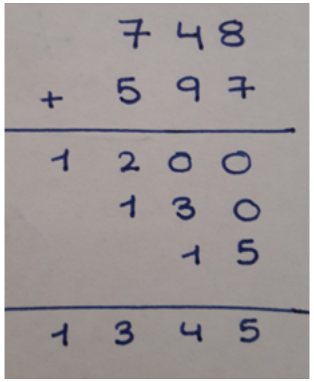
Exercise

* Fill in the boxes:

$$
\begin{array}{cccccc}
 & 7 & \square & 8 & 0 & 2_{(9} \\
- & 5 & 5 & \square & \square & 4_{(9} \\
\hline
 & 1 & 8 & 0 & 2 & \square_{(9}
\end{array}
$$

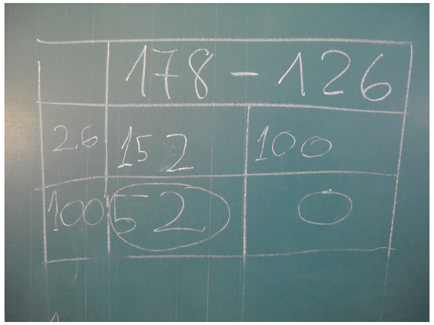
Alternative algorithms?

* Can you find a substraction algorithm analogous to this one?



* ABN: substraction algorithms

437 - 248		
QUITO	QUEDAN POR QUITAR	RESTAN
235	13	202
10	3	192
3	0	189



* Compute these substractions using ABN algorithms.

a) $104 - 49$ b) $824 - 347$

* Do you think it is interesting to compute substractions in base b using ABN algorithms? Why?

Mental calculation. “Natural” calculation?

* It is part of the curriculum.

  It most cases, not enough time is devoted to it.

* It is very important to properly develop number sense.

* We will practice in some problem sessions doing some “number talks” (Joe Boaler):

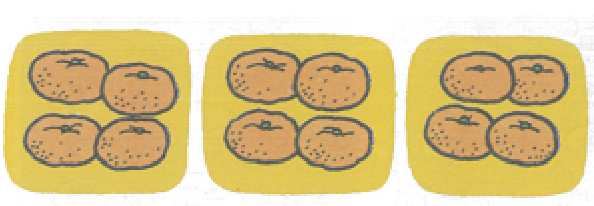
a) $89 + 43$ b) $56 + 35$ c) $83 - 28$ d) $79 - 42$

* The objective is not to memorize some tricks, but to develop personal strategies.

* Important: don’t try to mimic traditional pen and pencil algorithms!

Multiplication

* How should it be introduced?

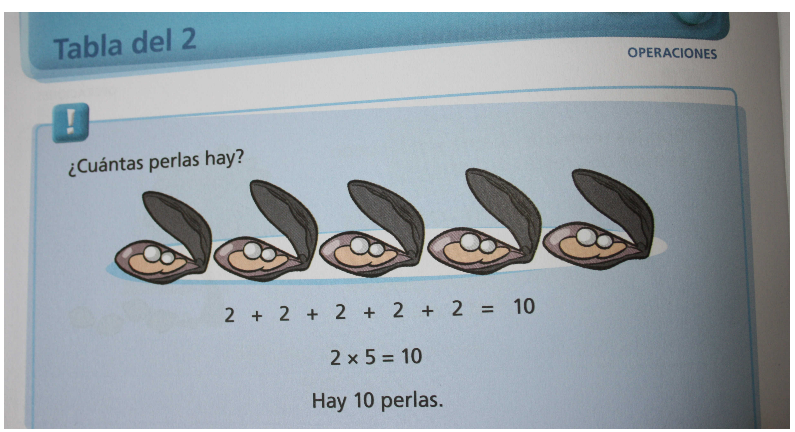


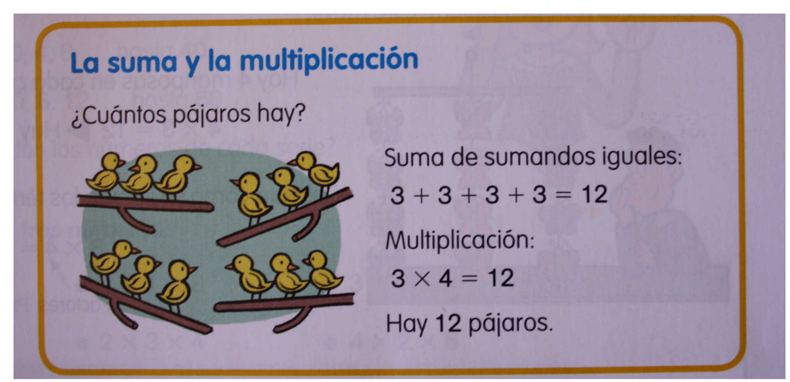
We have 3 dishes with 4 doughnuts in each dish. How many doughnuts are there in total?

* There are $4 + 4 + 4$ doughnuts.
  There are 3 times 4 doughnuts.

$$\text{3 times 4} \quad \leftrightarrow \quad 3 \times 4\ ?$$

* In (Spanish) textbooks





Multiplication

* What happens if we assume that:
  a) 3×4 means 3 times 4.
  b) in the times table for 2, we “count by two”.

* The traditional order for times tables does not match this proposal for the definition of multiplication.

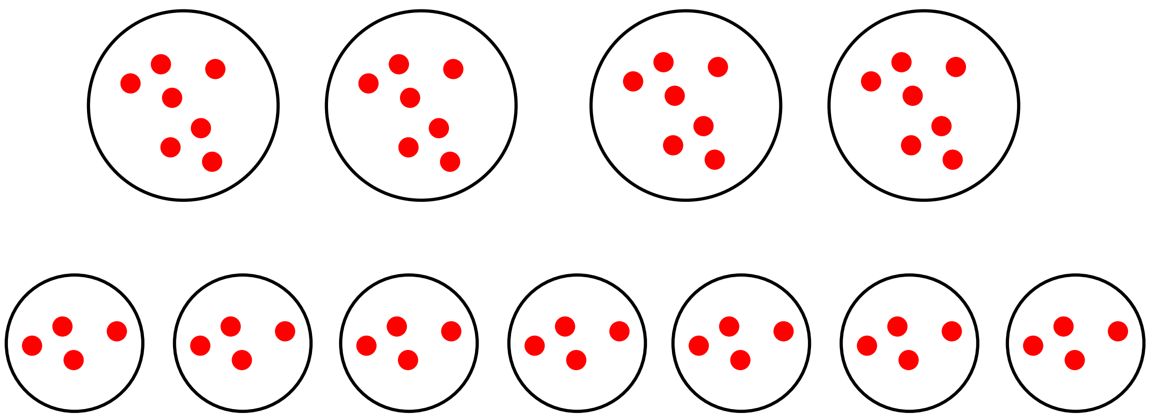
* In English, both orders can be found:
  - http://www.youtube.com/watch?v=tRMoBDyb9Jg
  - http://www.youtube.com/watch?v=vzXcI49jdV0

* More in Didactics of Math.

Multiplication properties

* Conmutative law: $a \times b = b \times a$.

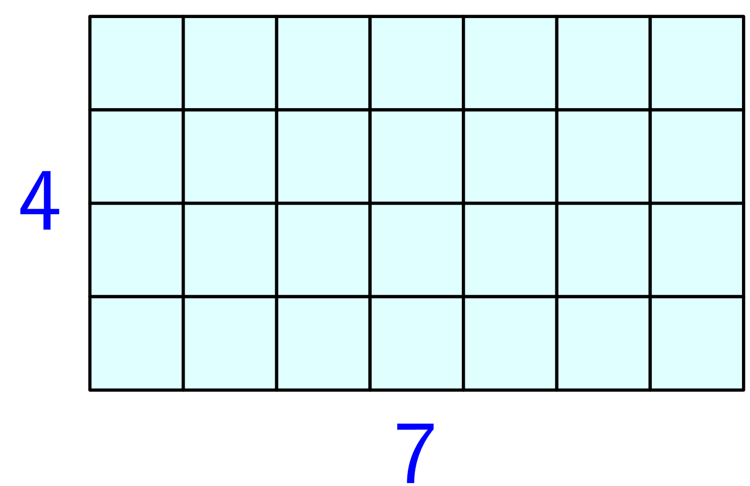
* It is not obvious that 4 times 7 is the same as 7 times 4 ....



4 times 7 $\leftrightarrow$ 4×7

7 times 4 $\leftrightarrow$ 7×4

* Geometry can be a perfect tool to explain some basic facts.



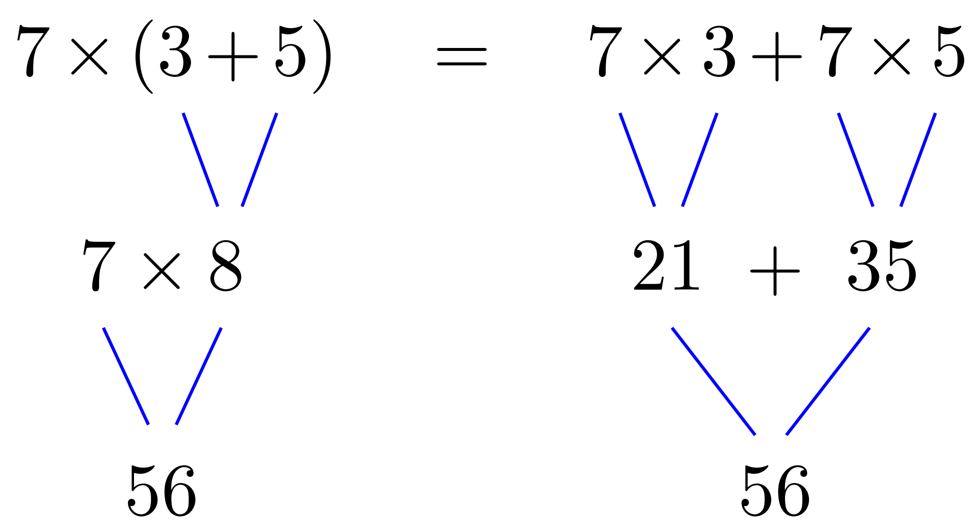
Distributive law

* Distributive law:

$$a \times (b + c) = (a \times b) + (a \times c)$$
$$(a + b) \times c = (a \times c) + (b \times c)$$

* Does it make sanse in elementary school?

* In textbooks ...



What for?

Distributive law

* In basic mathematics (primary and secondary school) it is used in two different situations:

  i) algebra: $2(x+3) = 2x + 6$

  ii) mental calculation (natural calculation):

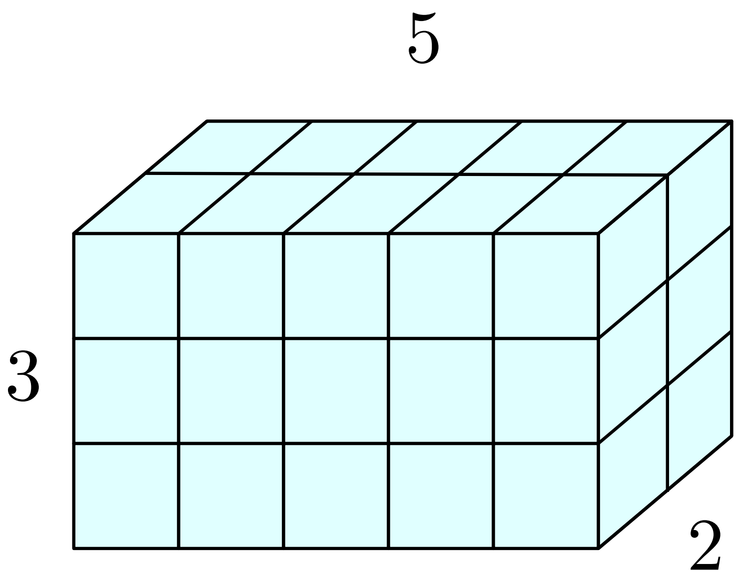
$$13 \times 8 = (10+3) \times 8 = 80 + 24 = 104$$

* Do you know why the standard algorithm works?

$$\begin{array}{r} 3\,8\,2 \\ \times \quad 2\,6 \\ \hline 2\,2\,9\,2 \\ 7\,6\,4 \\ \hline 9\,9\,3\,2 \end{array}$$

Associative law

- Associative law: $a \times (b \times c) = (a \times b) \times c$



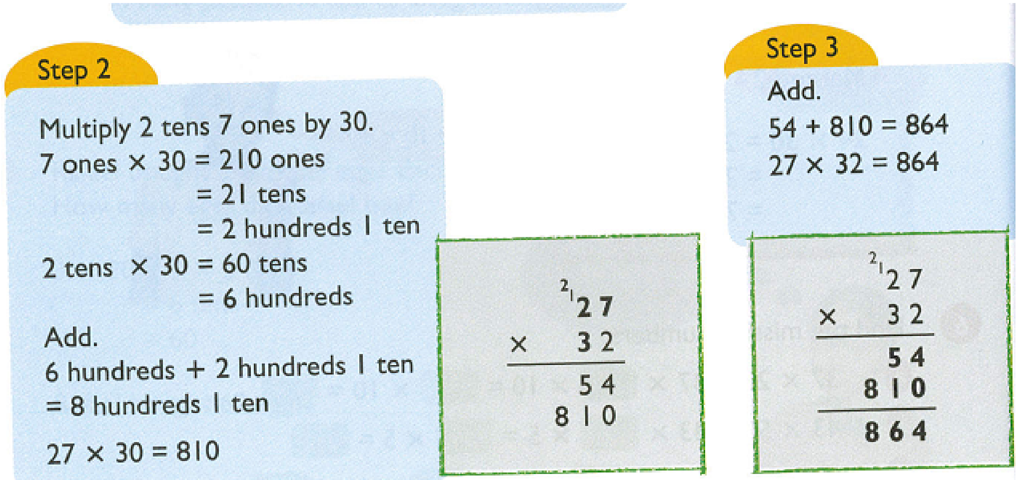
$$2 \times (3 \times 5) = (2 \times 3) \times 5$$

- A primary school problem that can be used as an example: We have two bags, inside each bag there are three boxes, and inside each box there are four sweets. How many sweets do we have in total?
- A frequent mistake: $2 \times (3 \times 5) =$

Alternative algorithms for multiplication

* The standard, properly explained

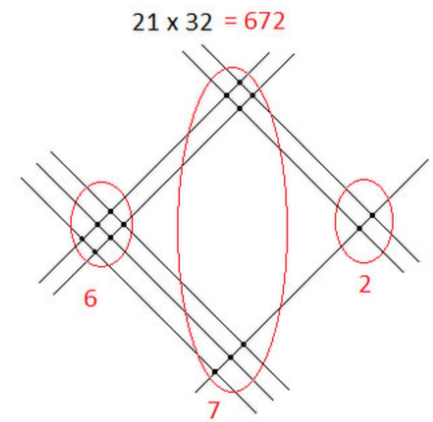
  (Singapore, Primary-4)



MULTIPLICANDO DESCOMPUESTO EN UNIDADES	MULTIPLICADOR POR DECENAS	MULTIPLICADOR POR UNIDADES	PRODUCTOS PARCIALES	PRODUCTO ACUMULADO
	70	4		
200	14000	800	14800	
80	5600	320	5920	20720
5	350	20	370	21090

ABN algorithm

285×74



Mayan algorithm

21×32

Exercises

* Explain why ABN and Mayan multiplication algorithms work.

* Compute the following multiplication using these algorithms, and think what advantages and drawbacks you think they have (also with respect to the traditional one).

$$45 \times 36 =$$

* Knowing that $652 \times 68 = 44336$, use the distributive law to compute the following multiplication, without further long calculations:

$$662 \times 68 =$$

* Could you compare the following products, without computing them?

$$835 \times 374 \quad \begin{matrix} < \\ > \end{matrix} \quad 834 \times 375$$

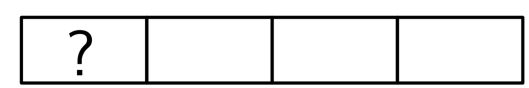
Division

* First comment: it is important to distinguish between the meaning of division and the algorithms for division.

* If a 6 year old kid has 8 candies and want to share them (equally) with a friend, will he be able to to it?

* This idea of equal sharing is the best one to introduce division: we refer to it as se partitive division.

* If 20 candies are put in 4 equal bags, how may candies will contain each bag?



Division

* There exists another interpretation of division: If 20 candies are put in bags and each bag contains 5 candies, how many bags there will be?

5	$\ldots ? \ldots$

* This is the quotative division (barely studied).
Related to measure: how may times does 5 fit in 20?

Division - Two different meanings

* Two observations:
  i) An easy way of distinguish them: think about how a 6 year old kid would solve the problem.

  ii) In quotative division, the divisor can be a rational number: A group of frieds buys 6 pizzas, and share them equally. If each friend eats $2/3$ of a pizza, how may friends ared there in the group?

1. Answer the question using your knoweledge about fractions.
2. Try to find an argument that you could use to explain the solution to a 9 years old student (he/she understads the concept of fraction, but does not know the arithmetic).

Division

- A good way to understand both types of division:

  Make up two problems (one of each type) whose solution contain the division $72 \div 6$.

- Another important idea, worth to spend some time with it is the division as inverse of multiplication:

  Because $5 \times 4 = 20$, we have $20 \div 5 = 4$ and $20 \div 4 = 5$.

- Why división by 0 cannot be defined?

$$5 \div 0 = ? \quad \leftrightarrow \quad ? \times 0 = 5 \qquad \text{no solution}$$

$$0 \div 0 = ? \quad \leftrightarrow \quad ? \times 0 = 0 \qquad \text{infinite solutions}$$

- More in didactics.

Division with remainder

* Given two natural numbers D (dividend) and d (divisor), there exist unique natural numbers q (quotient) and r (remainder) such that

$$D = q \times d + r \quad \text{and} \quad 0 \leq r \leq d - 1$$

.

* Basic idea of every division algorithm:

  Approximate (from below) the dividend using multiples of the divisor.

$$16 = \square \times 3 + \square$$

$$\wedge$$

$$3$$

Problems

* Write two problems with data 27 and 4. In one of them, the solution has to be 6 and in the other one the solution has to be 7.

* An aspect that is overlooked frequently: problems where the remainder is relevant.

  The journey of an astronaut lasted 505 hours. If he took off at 8 am, what time was it when he landed?

* Knowing that $635 \times 97 = 61595$, explain how you could compute quotient and remainder of 61695 divided by 97 without any further long computation.

* If you know that when 64757 is divided by 439 the quotient is 147 and the remainder is 224, what are the quotient and remainder when 64757 is divided by 147?

Division in $\mathbb{N}$

* Standard algorithms for division:

“Extended” algorithm

$$\begin{array}{rrr|ll}
6 & 4 & 0 & 2 & 3 \\ \cline{4-5}
-4 & 6 & & 2 & 7 \\ \cline{1-2}
1 & 8 & 0 & & \\
-1 & 6 & 1 & & \\ \cline{1-3}
 & 1 & 9 & &
\end{array}$$

“Standard” algorithm (in Spain) (“compressed”)

$$\begin{array}{rrr|ll}
6 & 4 & 0 & 2 & 3 \\ \cline{4-5}
1 & 8 & 0 & 2 & 7 \\
 & 1 & 9 & &
\end{array}$$

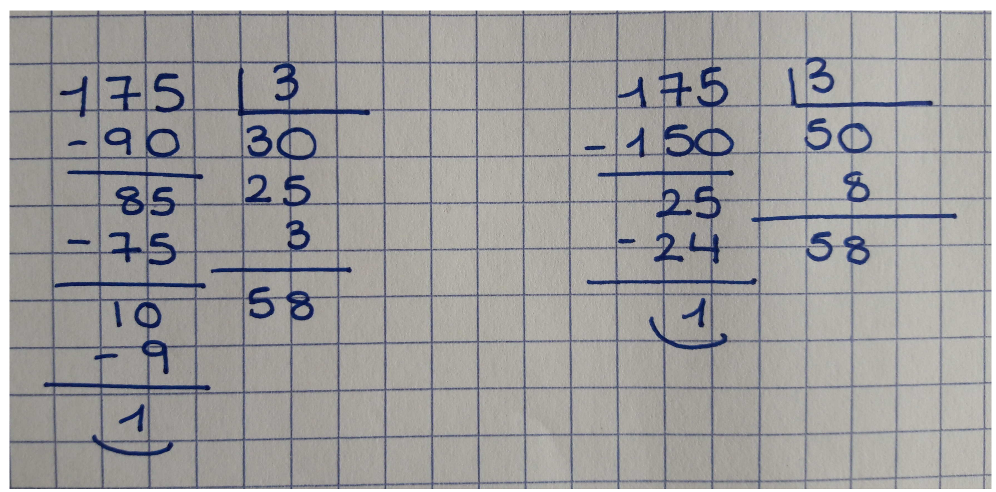
Alternative algorithms?

ABN

DIVIDENDO	DIVIDENDO RESULTANTE	COCIENTES PARCIALES
		6
7899	6000	1000
1899	1800	300
99	60	10
39	36	6
3		
7896 : 6 =		1316

		: 57
19.368	17.100	300
2.268	1.710	30
558	513	9
45		339

A proposal



Exercises

* Compute the following divisions using the last proposed algorithm. Check that you understand each step of the process.

$$97 \div 4 \qquad\qquad 835 \div 37$$

* Something to think about: Besides consider which division algorithm is more convenient, it would be worth to think on the real value of division algorithms.

  In a lot of countries, divisors with two (or more) digits are not considered in the primary school curriculum.

Exercises

1. What happens whith quotient and remainder when dividend and divisor are multiplied (or divided) by the same number?

2. Knowing that $4185 = 45 \times 93$, find (without making the division) quotient and remainder when 41862 is divided by 930.

3. Find three 4-digit numbers that give 7 as remainder when they are divided by 19.

4. Find the smallest number that is bigger than 300 and has 7 as remainder when divided by 29.

The calculator (and other devices)

* It is considered in the curriculum, and it should be integrated in math classes.

  if nothing else, at least to avoid situations like this one:
  https://www.youtube.com/watch?v=zclITKd4ivQ

* Use your calculator to compute your body mass index

  (Height in meters)

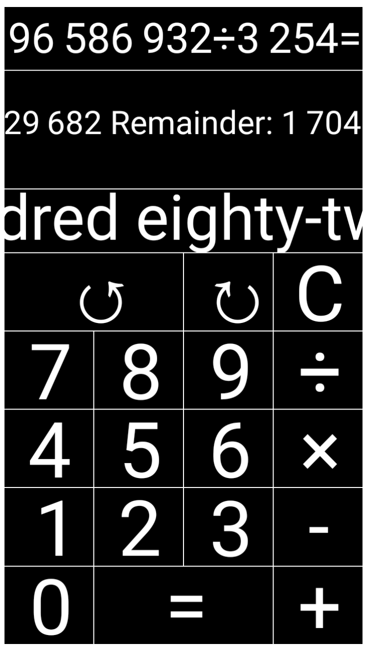
* Two different considerations:
  (1) it can be used to avoid complicated or long calculations, or to check results.
  (2) it can be used to design some learning activities.

The calculator (and other devices)

* Division with remainder in a standard calculator.

$$29374 \div 387 \approx 75,902$$

* Other alternatives:



Whole calculator (free)

Calculadora natural (73 c)

(Android)

https://www.wolframalpha.com/

Example of a learning activity

* Broken calculators.

Freudenthal Institute:
http://www.fisme.science.uu.nl/toepassingen/03363/