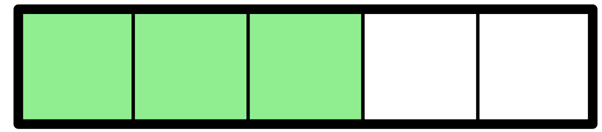
Lesson 2: Fractions and proportionality

- ⋆ Fractions
- ⋆ Rational numbers
- ⋆ Decimal numbers
- ⋆ Ratio and proportionality
- ⋆ Percentages

Fractions: an object, several interpretations

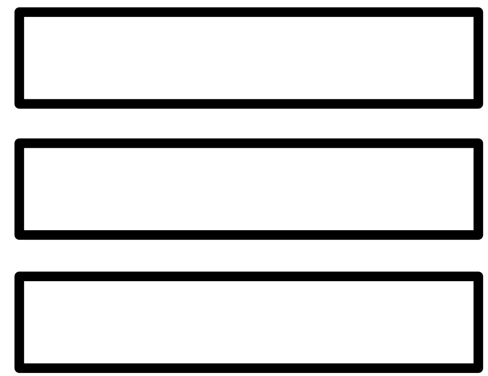
(1) A part of the whole



We have colored $3/5$ of ...

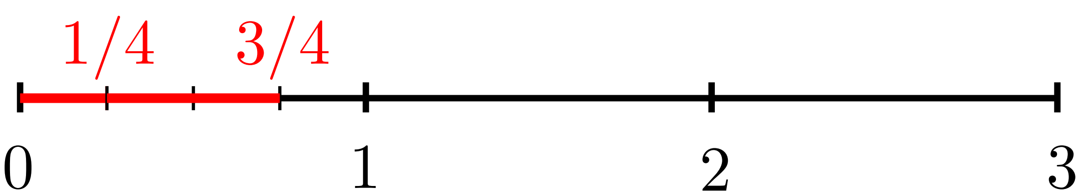
(2) Share (division)

We want to share 3 candy bars among 5 kids.
How much chocolate eats each kid?



(3) A point in the number line (a number)

$$¿\ \frac{3}{4}\ ?$$

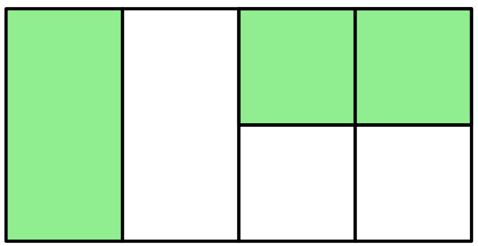


The denominator fixes the unit

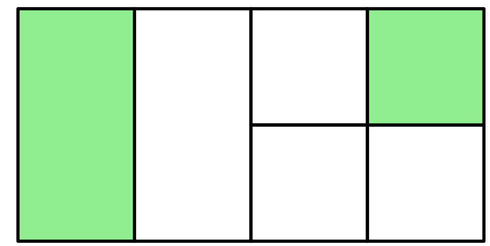
The numerator is the number of units you take

Some examples

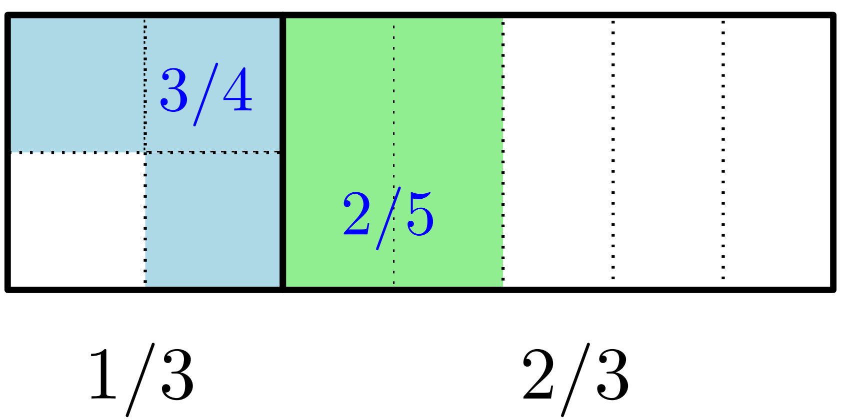
* What fraction of each figure is colored?



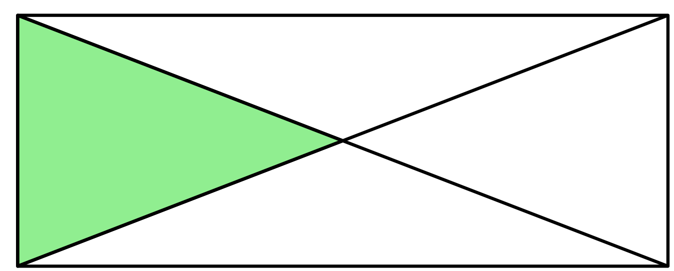
(a)



(b)



(c)



(d)

Some examples

* I have eaten $1/3$ of the chocolates in a box and there are 12 left. How many chocolates had the box when it was full?

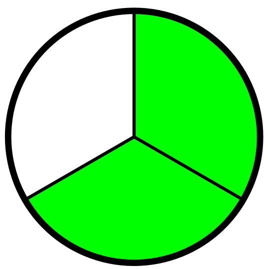
* Paul read $2/5$ of the pages of a book on Monday, on Tuesday he was busy and he was able to read only $1/3$ of the pages that he read on Monday, and on Wednesday he was free so he was able to read 140 pages and finish the book. How many pages does the book have?

Definition of fraction

* A fraction is an expression with the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

numerador

denominador

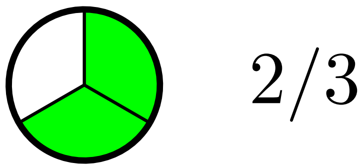


$\frac{2}{3}$

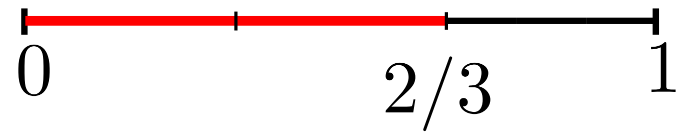
Part of a whole

Quantity

Point on the number line



Part of a whole



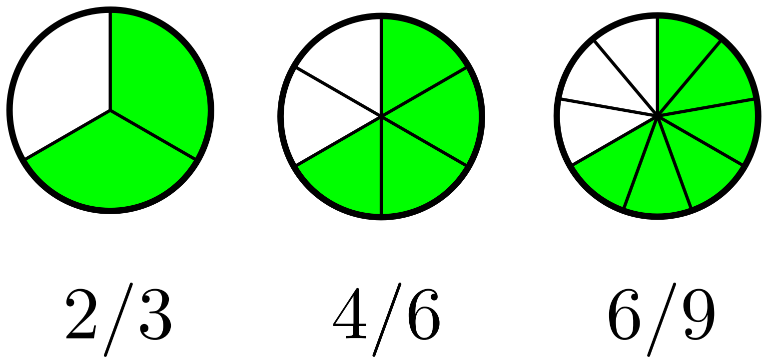
Point in the number line

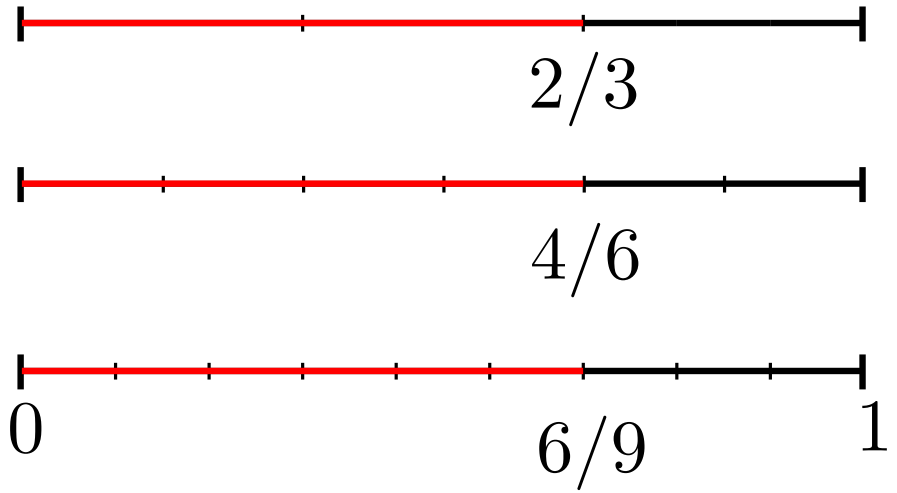
* Understanding both interpretations is necessary, and each of them has its own advantadges.

  How to combine them is an important issue in didactics of mathematics.

∗ Fractions such $2/3$, $4/6$, $6/9, \ldots$ represent the same quantity.

We will say that they are equivalent fractions.





∗ Def: We say that a number is rational if it can be expressed as the quotient of two integers, i.e., if it can be expressed as a fraction.

The set of rational numbers is denoted by $\mathbb{Q}$.

Fractions and the number line

* The big advantadge of interpreting fractions on the number line is that it shows, right from the beginning, that fractions are an extension of the already known numbers.

* Exercices like

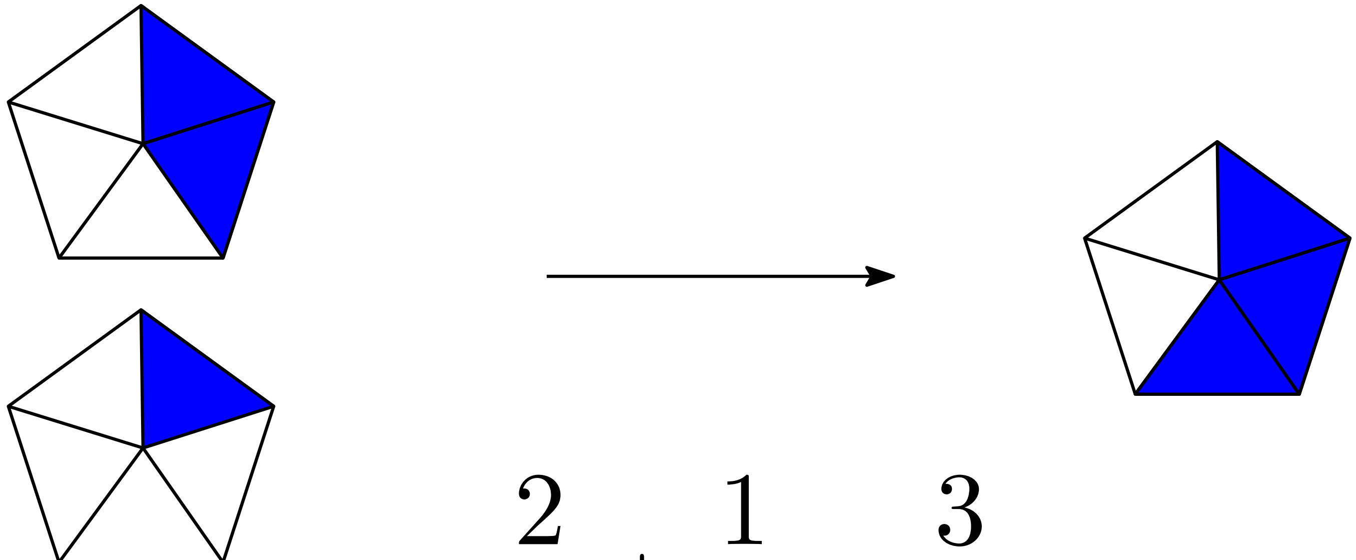
  Represent on the number line 1, 2, $6/7$ y $13/5$

  help to develop the understanding of fractions.

* Improper fractions are not a problem.

Addition of fractions

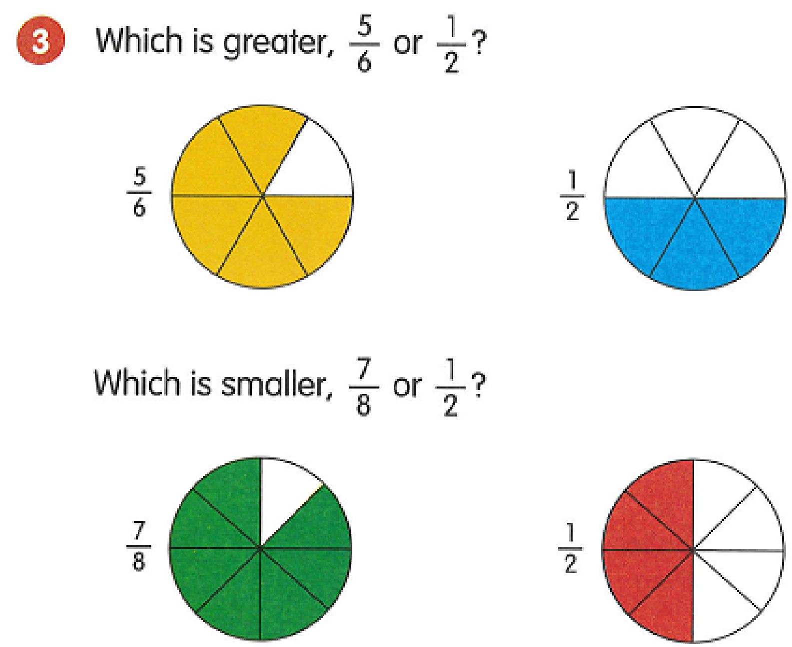
* Same denominator: The most common approach may have a problem?



$$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$$

Equivalent fractions - Comparing fractions

* If we want kids to understand, and not only memorize, we have to avoid "recipes" and introduce examples as in the figure.



* Only after that ..
  1. equal denominator.
  2. equal numerator
  3. by comparison with a reference
  4. general case: common denominator

* Exercice. Compare the following fractions:

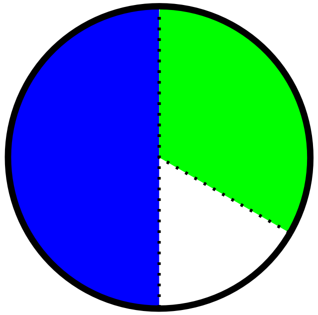
(a) $\frac{6}{7}$ and $\frac{7}{8}$ (b) $\frac{11}{23}$ and $\frac{22}{43}$ (c) $\frac{826}{825}$ and $\frac{1223}{1222}$

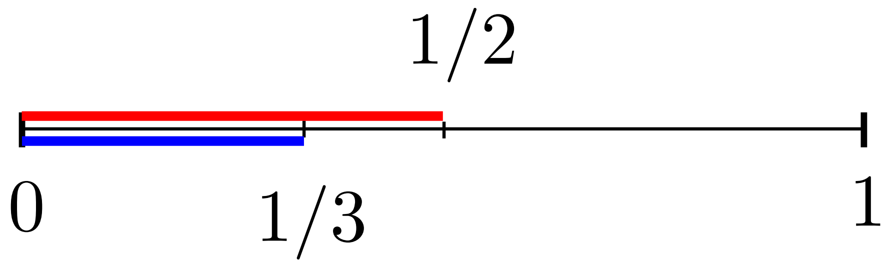
Addition of fractions, unlike denominators

- If previous concepts have been properly understood, we can propose the problem:

$$\text{How much is } \frac{1}{2} + \frac{1}{3}?$$

- For a kid working on this “problem” (with a proper model) it is not difficult to understand the impossibility of adding up fractions with different denominators.



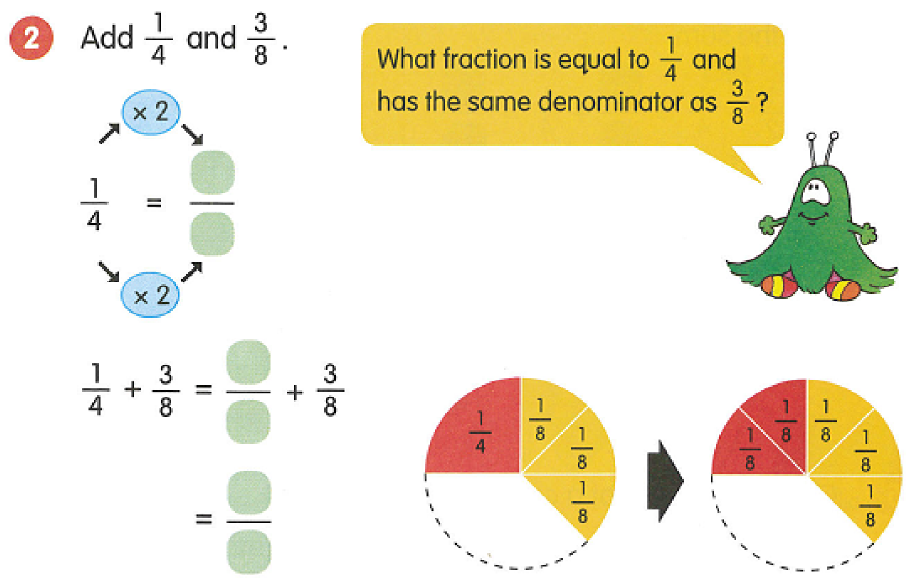


Equivalent fractions. Addition and substraction

* Once the concepts of fraction and equivalent fractions are properly understood, addition and substraction should be easy to understand.

  a) Fraction with different denominators cannot be added (nor substracted).
  b) Therefore, before adding or substracting we have to find equivalent fractions with the same denominator.

* Instead of giving the procedure (you have to get same denominator and ...) you can consider examples as in the figure (scaffolding and zone of proximal development – Vygotsky)



Improper fractions, mixed numbers

* A fraction such as $17/12$ (in general, fractions a/b where $a \geq b$) are usually called improper fractions and can also be represented as mixed numbers:

$$\frac{17}{12} = 1\,\frac{5}{12} = 1 + \frac{5}{12}$$

* Strongly related to division with remainder:

$$D = q \times d + r \quad \rightarrow \quad \frac{D}{d} = q + \frac{r}{d}$$

* It is important to keep in mind that if option 1 has been used (part of a whole), improper fractions are a nontrivial generalization: what does eight sevenths of something mean?

Multiplication of fractions

* If we think only in the algorithm, multiplying fractions is much easier than adding them.
  Nevetheless, the concept is much more complicated.

* A good possibility is to generalize from integers, as follows:
  - 2×18 is "two times 18"
  - $\frac{1}{3} \times 18$ is "one third of 18", i.e., $\frac{18}{3}$

* It is worth to pay attention here to a very important relation between multiplication and divisionr:

  multiplying by $\frac{1}{n}$ is the same as dividing by n

Multiplication of fractions

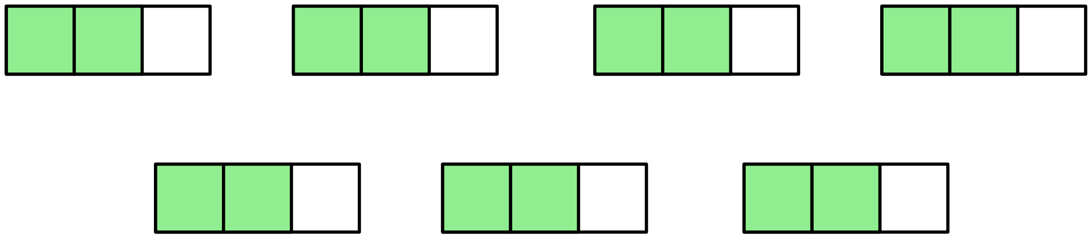
* Once that we understand the mutiplication of $1/n$ by an integer (as a division by n) we can multiply $1/n$ by other fraction:

a) $\frac{1}{2} \times \frac{8}{15} = \frac{4}{15}$ b) $\frac{1}{2} \times \frac{7}{15} = \frac{1}{2} \times \frac{14}{30} = \frac{7}{30}$

* Fraction of a set: $\frac{2}{3}$ de 7

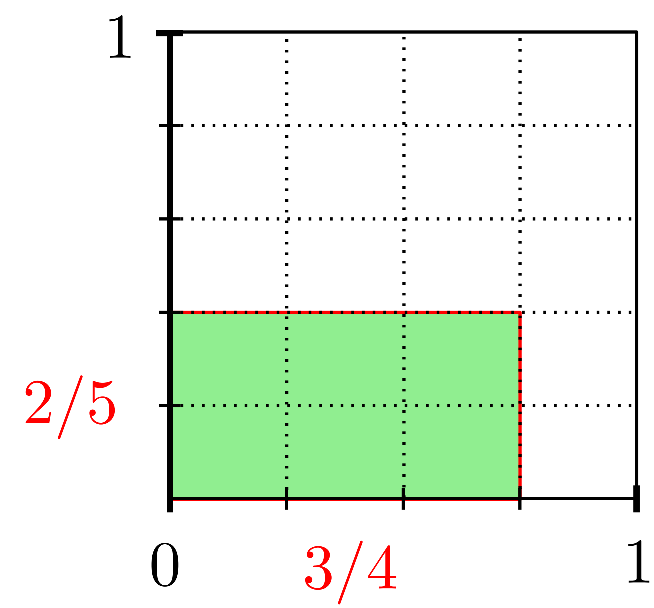
$$\frac{2}{3} \times 7 = 2 \times \left(\frac{1}{3} \times 7\right) = 2 \times \frac{7}{3} = \frac{14}{3}$$

Grafically:

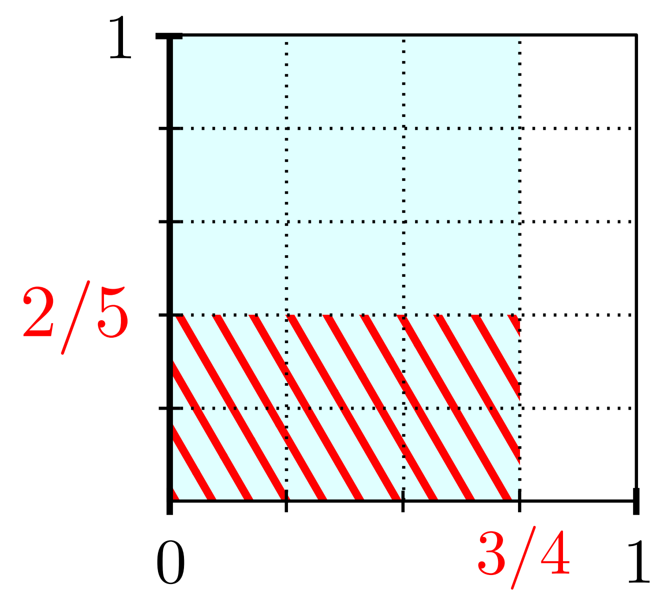


Multiplication of fractions. Area model

$$\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}$$

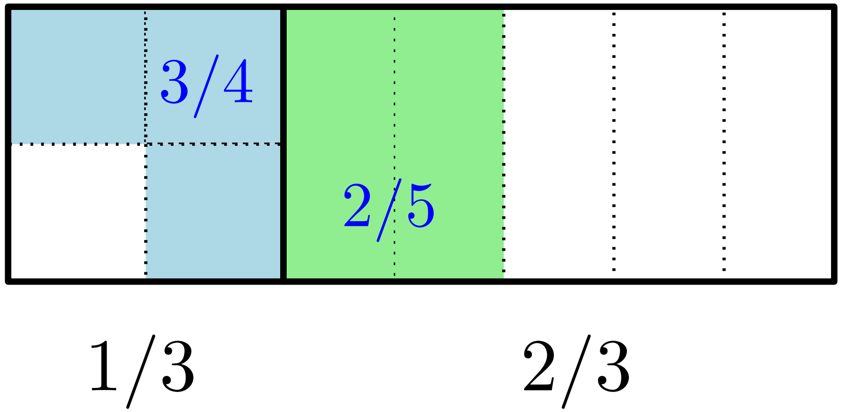


Here it can also be seen that $\frac{2}{5} \times \frac{3}{4}$ means $2/5$ of $3/4$.



Exercises

* What fraction of the total area has been colored?



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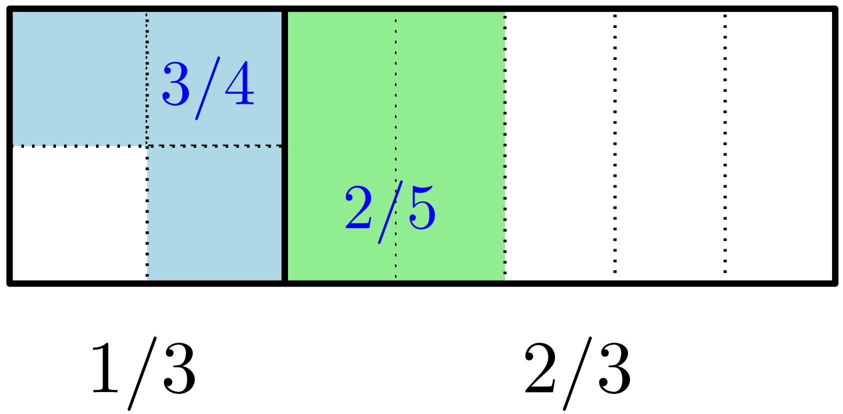


* If I pour 6 glasses in a bottle and $1/4$ of each glass is alcohol, what fraction of the liquid in the bottle will be alcohol?

Division

* First, something that I think is not a good alternative.

División de fracciones

Ester tiene 2 kg y medio de almendras.
Las reparte en bolsas de un cuarto de kilo cada una.
¿Cuántas bolsas puede preparar?

Almendras — $2\frac{1}{2}$ kg ▸ ▸ $\frac{5}{2}$ kg

Bolsas de $\frac{1}{4}$ kg — 1 kg = 4 bolsas ▸ ▸ 10 bolsas de $\frac{1}{4}$ kg

Calcula cuántos $\frac{1}{4}$ hay en $\frac{5}{2}$, es decir, divide $\frac{5}{2}$ entre $\frac{1}{4}$

- El numerador es el producto del numerador de la primera fracción por el denominador de la segunda.
- El denominador es el producto del denominador de la primera fracción por el numerador de la segunda.

$$\frac{5}{2} : \frac{1}{4} = \frac{5 \times 4}{2 \times 1} = \frac{20}{2} = 10$$

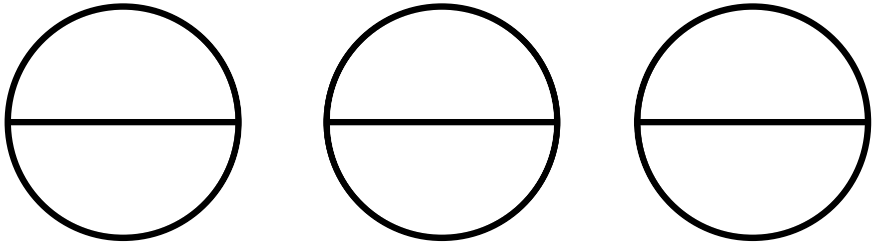
Puede preparar 10 bolsas de un cuarto de kilo.

Para dividir dos fracciones, se multiplican sus términos en cruz.

First examples

* If we want kids to understand the concept, we have to start with simple examples:

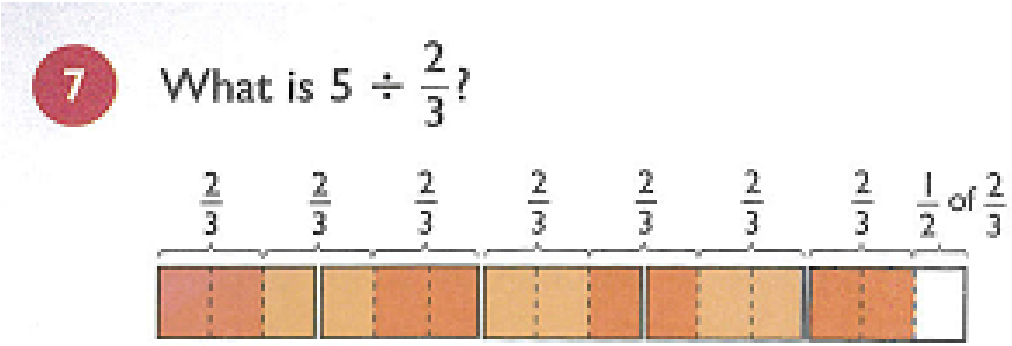
  A group of friends buy 3 pizzas and eat half a pizza each. How many friends are there in the group?



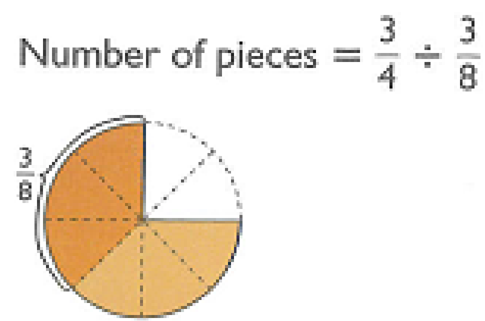
$$3 \div \frac{1}{2} = 3 \times 2 = 6$$

* With examples as this one it can be understood that dividing by $1/n$ is the same as multiplying by n.

An example (Singapore, 6°)

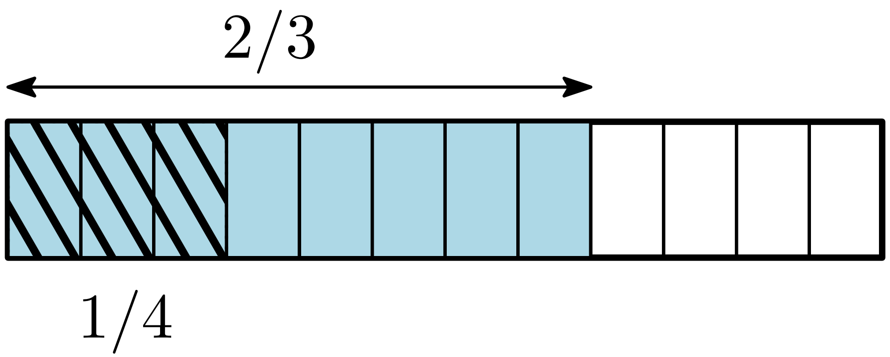


* How many $\frac{2}{3}$ are there in two units?
* In one unit there are [ ] $\frac{2}{3}$.
* So, in 5 units there are [ ] $\frac{2}{3}$.
* And the general case (dividing two fractions)



An alternative approach: common denominator

* $\frac{2}{3} \div \frac{1}{4} = \frac{8}{12} \div \frac{3}{12} = \frac{8}{3} = 2 + \frac{2}{3}$



* Two options for dividing fractions:

1. $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8}$
2. $\frac{3}{4} \div \frac{2}{5} = \frac{15}{20} \div \frac{8}{20} = \frac{15}{8}$

* Pros and cons?

Problems

* We have a barrel with 350 liters of water, and we want to fill up bottles with capacity $3/8$ of liter. How many bottles will we fill up?

* A person leaves $2/3$ of his heritage to his only daugther, $4/5$ of the remaining part to an old uncle, he has to pay as taxes $1/20$ of the total heritage and he donates the rest, 12000 euros, to charity. What was the total value of his heritage?

* Exercise: Compute and express as an irreducible fraction

$$\frac{1}{3}+\frac{2}{3}\times\frac{4}{7}\times\frac{9}{8}-2\times\left(\frac{1}{12}-\frac{7}{3}\right)-1$$

Order in $\mathbb{Q}$

* Order in $\mathbb{Q}$ is defined in the same way as in natural numbers: given two rational numbers a and b, we say that $a < b$ if $b - a > 0$.

* Properties of inequalities:

  a) If $a < b$ then $a + c < b + c$ (for every rational number c).

  b) If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.

  c) If $a < b$, then $-a > -b$.

  Therefore, if $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$

* Exercise: Find all rational numbers x for which $\dfrac{2}{3} - x < \dfrac{7}{5}$

Order in $\mathbb{Q}$

* Rational numbers are "dense": in $\mathbb{Q}$ the "next number" does not exist.

  Remark: between any two rational numbers there exist an infinite number of rational numbers.

* But there are numbers that are not rational:
  Theorem: $\sqrt{2}$ is not a rational number.

* Last remark: The set of rational numbers is "countable" (we can make a list with all of them).

  (Of course, the list is infinite).

Problem

* We have a bathtub with two faucets. It takes 1 hour for the hot water faucet to fill in the bathtub, while it takes 30 minutes for the cold water faucet. If both faucets are opened at the same time, and the flow in each of them is the same as when they were opened alone, how long will it take for the bathtub to be full of water?