Lesson 2-3: Proportionality and percentages

* A ratio is a relation between two quantities. Ex:
  a) in a bag containing white balls and black balls the ratio between white and black balls is 2 to 7.
  b) in an exam, the ratio between student that pass and students that fail is 4 to 3.

* The same notation is commonly used for ratio and fractions, but concepts are different. Two important differences:
  1. A ratio can involve heterogeneous magnitudes (with different units). Ex: a car consumes 6 liters/100 km.
  2. A ratio can be an irrational number.
     Example: the ratio between the length of a circle and its diameter.

Ratio - Fraction

* Imagine that we have a bag with red balls and blue balls. We know that for each 3 red balls there are 2 blue balls.

  We write that the ration between red and blue balls is $3 : 2$ or also $3/2$.

* If we know that the ration between red and blue balls is $3 : 2$, what fraction of the total number of balls are red?

* In common language it is usually said what proportion are red?

  This use is not precise, and we will try to avoid it here. In mathematics, a proportion is something that we are going to study now.

Proportions

* A proportion is an equality between two ratios.
* Example: in some exam the ratio between students that pass and students that failed is 4 to 3. If 81 students failed, how many students passed the exam?

$$\frac{4}{3} = \frac{x}{81} \quad \rightarrow \quad 3x = 4 \cdot 81 \quad \rightarrow \quad x = \frac{4 \cdot 81}{3} = 108$$

* And yes ... it can be also interpreted as the classic “regla de tres”:

$$\begin{array}{ccc} 3 & \rightarrow & 81 \\ 4 & \rightarrow & x \end{array}$$

The drawback of “la regla de 3”

4. EL RAZONAMIENTO DE LA REGLA DE TRES

Con la expresión “regla de tres” se designa un procedimiento que se aplica a la resolución de problemas de proporcionalidad en los cuales se conocen tres de los cuatro datos que componen las proporciones y se requiere calcular el cuarto. Aunque aplicado correctamente el razonamiento ¿supone una cierta ventaja algorítmica? en el proceso de solución, ya que se reduce a la secuencia de una multiplicación de dos de los números, seguida de una división por el tercero, con frecuencia muchos alumnos manipulan los números de una manera aleatoria y sin sentido de lo están haciendo. En cierto modo el algoritmo les impide comprender la naturaleza del problema, sin preocuparse de si la correspondencia entre las cantidades es de proporcionalidad directa, inversa, o de otro tipo. La regla de tres se llega a aplicar de manera indiscriminada en situaciones en las que es innecesaria o impertinente. or even worst, a mistake ...

From Godino et al: Proporcionalidad y su didáctica para maestros (p. 425), http://tinyurl.com/8kxkr7r

* A funny video: https://youtu.be/wkJrysJhU7s

Examples

* Problem: If 3 tickets for the movies cost 21 euros, how much do 5 similar tickets cost?

* Method 1:

  3 tickets cost 21 euros $\rightarrow$ each ticket costs 7 euros.

  Therefore, 5 tickets cost 35 euros.

  Reduction to unity

* Method 2:

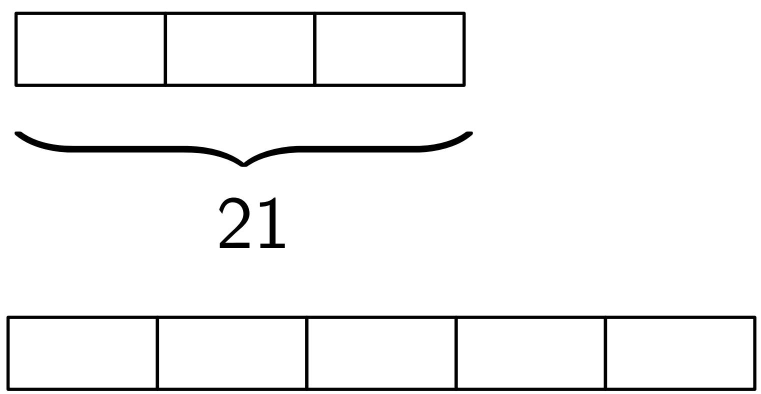
$$\left.\begin{array}{ccc} 3 & \rightarrow & 21 \\ 5 & \rightarrow & x \end{array}\right\} \quad \Rightarrow \quad x = \frac{21 \cdot 5}{3}$$

  Regla de tres

Examples

* Problem: If 3 tickets for the movies cost 21 euros, how much do 5 similar tickets cost?

  An alternative: The bar model (Singapore).

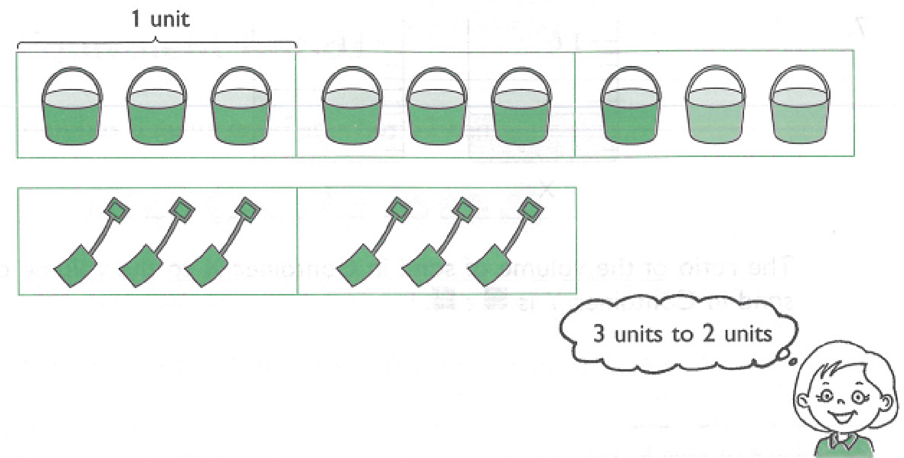


* At least, a good auxiliary alternative.

  Main advantadge: visualization helps to understand.

Some examples from Singapore textbooks

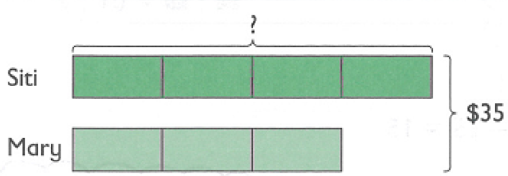
* Introducing ratio in Grade K-5



The ratio of the number of buckets to the number of shovels is ____ : ____.

* And then (also in Grade K-5)

6. Siti and Mary shared $35 in the ratio 4 : 3. How much money did Siti receive?



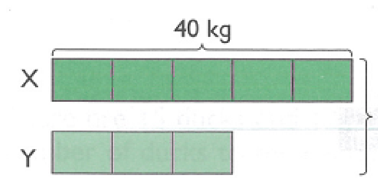
7 units = $35

1 unit = $____

4 units = $____

Siti received $____.

7. The ratio of the weight of Package X to the weight of Package Y is 5 : 3. If the weight of Package X is 40 kg, find the total weight of the two packages.



5 units = 40 kg

1 unit = ____ kg

8 units = ____ kg

The total weight is ____ kg.

Some examples from Singapore textbooks

* Finally, in Grade K-6:

2 The mass of potatoes used by Mrs Wee in her cooking was $\frac{5}{2}$ of the mass of carrots used. She used 9 kg more potatoes than carrots.

a Find the ratio of the mass of potatoes used to the mass of carrots used to the total mass of both ingredients.

b What fraction of the total mass of both ingredients was the mass of the potatoes?

c Find the total mass of both ingredients.

Directly proportional magnitudes

* Two magnitudes are son directly proportional if their ratio is constant.

  Example: space and time in a constant speed movement.

$$\frac{e_1}{t_1} = \frac{e_2}{t_2} = \cdots = \boxed{v}$$

proportionality ratio

* Exercise: Study whether or not the following magnitudes are directly proportional:

  1. The side lenght of a square and its area.
  2. The mass of an object and its volume.
  3. The time spent by an object in free fall and the height of the fall.
  4. The side lenght of a square and its perimeter.

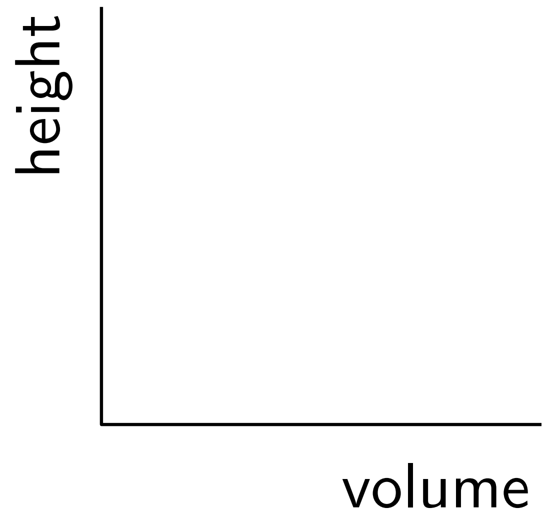
Directly proportional magnitudes

- In practice, things might be not that clear:
  Consider the cost of producing a given number of certain device and the number of devices. Are they proportional?

- An experiment (easy and instructive)
  Make a table with the amount of water poured in the following glasses (measured in cl, in spoons, ...) and the height that is reached in the glass. For which glass the sequences are proportional?

Proportional sequences

* If we represent data obtained from previous experiment, what do we get?



(1)

(2)

(3)

Problems

* An important example: the scale in a map.

  The distance between two points in a map with scale 1:20000 is 6.7 cm. How much is the real distance?

* The ratio between the dimensions of two rectangles is r. What is the ratio of their perimeters? And the ratio of their area?

* In a given exam, I got 7 points from a total of 13 points. What will be my grade?

* Eight men need 6 hours in order to cut 24 tree trunks. How much time will five men need in order to cut 10 tree trunks, if they work at the same speed?

  An alternative to "regla de 3 compuesta": reduction to unity.

Inverse proportion

* We say that two magnitudes x and y are inversely proportional if their product is constant.
* Examples:

  - speed and time in uniform motion:

  A car travels with constant speed. If the speed increases by $1/5$, how much does the time decrease?

  - presure and volume of a gas at constant temperature

$$PV = k \qquad \text{(Boyle-Mariotte's Law)}$$

  At constant temperature, the voumen of a gas duplicates. What is the change in the pressure?

Inverse proportion

* **Exercise:** If the lenght of a rectangle increases by $1/5$, how much should decrease the width so that the area remains constant?

$$A = b \times h$$

Problems

* A group of hikers has a supply of drinking water for 12 days. If during the trip the daily intake is 50% bigger than the forecast, when will they run out of water?

* We know that a tap with a given flow takes 4 to fill up a tank. If the flow decreases to $2/3$ of the original, how much will it take to fill up the tank?

* A ship goes from Naples to Barcelona and the journey takes 30 days. Another ship goes from Barcelona to Naples and the journey takes 20 days. At which point of the trip will the ships find each other? (We assume that both ships use the same route, and that each ship moves at a constant speed).

Percentage

* A percentage is a ratio whose denominator is 100.

  It is perhaps the most common mathematical concept in daily life.

* Examples:

  a) In an election in 2004 party A got 4323890 out of 11523876 votes, while in 2008 election it got 4387905 out of 11600399 votes.

  $$2004 \to 37.5\% \qquad\qquad 2008 \to 37.8\%$$

  b) Salary changes, price increments, sales, taxes ...

Elementary examples

1. Compute a given percentage of a given quantity.
2. Write a ratio as a percentage.

* Do we know how to reason about percentages?

  Assume that a given shop reduces the price of some trousers by 10% for the sale season, and that they increase again the price by 10% at the end of the sale season. The price of the trousers after both changes is:

  a) the same as the first price.
  b) bigger that the first price.
  c) smaller than the first price.
  d) bigger or smaller, depending on the initial price.

Elementary examples

* An employee got a 2% rise on her salary for two years in a row. After those two increments, the salary is:
  a) 4% bigger than the original one.
  b) More than 4% bigger than the original one.
  c) Less than 4% bigger than the original one.
  d) More or less than 4% bigger than the original one, depending on the original salary.
* What is bigger, 37% of 85 or 85% of 37?

Efficient computation of percentages

* Examples:

  a) The price of a 85 euros dress is reduced by 30%. What is the new price?

  b) John's monthly wage is 1250. He gets a 2% increase. How much will the new salary be?

  Do you know how to answer these questions with a single operation?

* Problem:

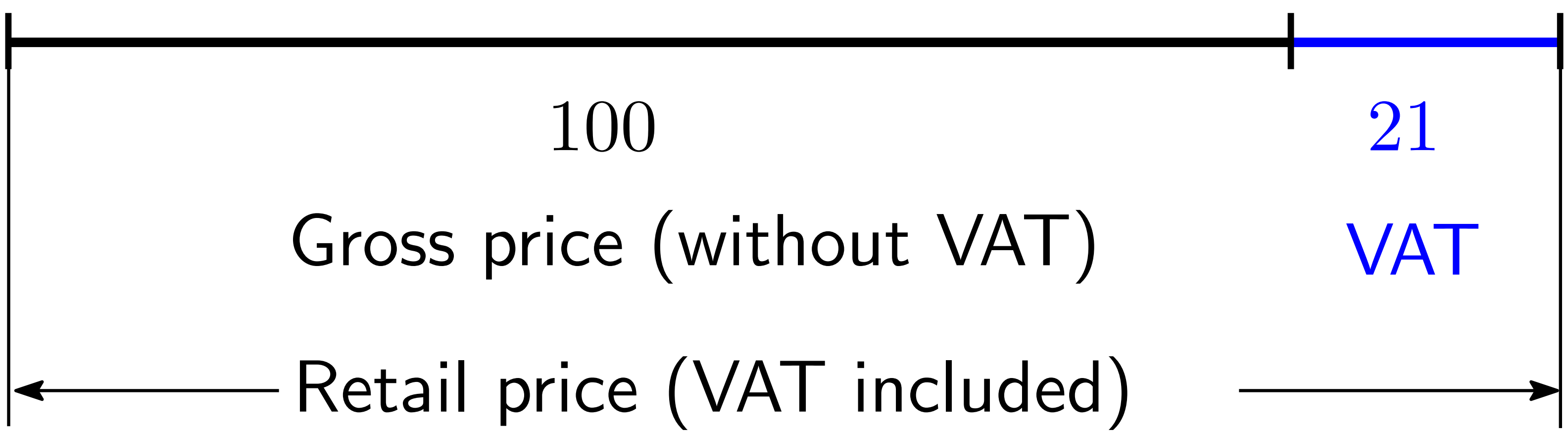
  Lisa started working for a company 10 years ago. Her salary was 900 euros and it has increased by 2% every year. How much does she get this year?

Taxes and percentages

* VAT (IVA)

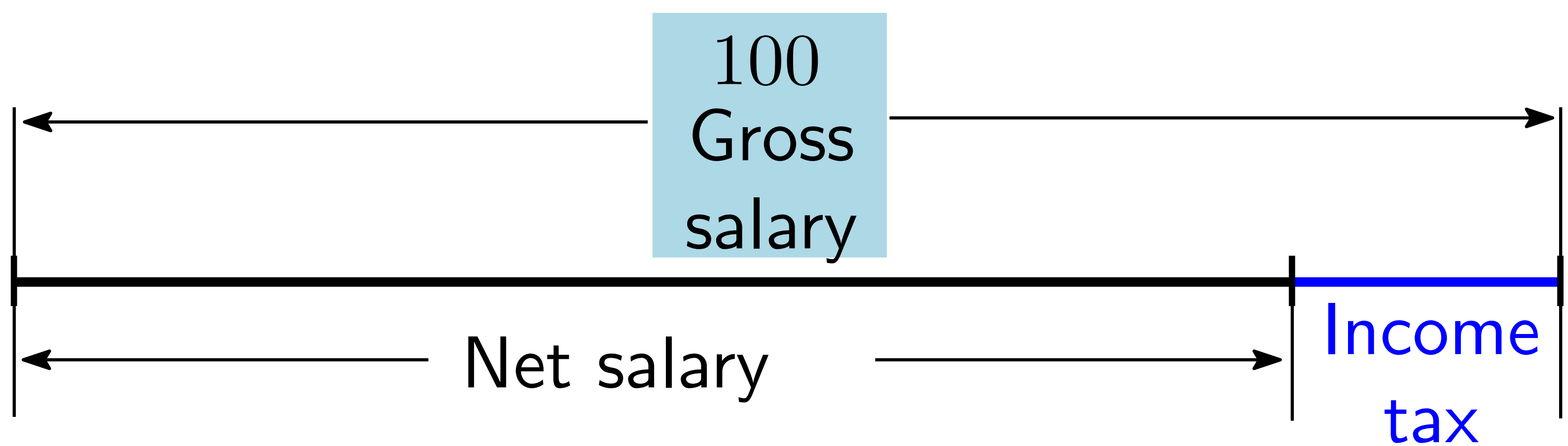
The retail price is obtained by adding VAT (for instance, 21%) to the gross price.

1. The gross price of a pair of sneakers is 55 euros. What is their retail price? (VAT is 21%)
2. The retail price of a suit is 120. How many euros will I pay as VAT (take VAT as 21%)?



Taxes and percentages

* Income tax (IRPF)
  The company pays to the employee a gross salary but a deduction is made (the income tax). The employee gets at his account the net salary.
  1. Mario has a monthly gross salary of 1250 euros and the income tax is 19%. How much is the net salary?
  2. Afther paying an income tax of 24%, Paula gets a net salary of 1420 euros. How much is her gross salary?



Problems

* I have payed 63 euros for a suit whose price was reduced by 30%. How much was the price before the reduction?

* John does not remember his salary during the year 2008, but he knows that from 2008 to 2009 he got a 3% increase and from 2009 to 2010 he got a 2% increase. If 2010 salary was 1345 euros, how much was his 2008 salary?

* I want to buy a car that is on sale, and the salesman allows me to choose how to compute the price:
  1. Option 1: the price is reduced by 15% and then the 21% VAT is added.
  2. Option 2: the 21% VAT is added and then the price is reduced by 15%.

  What option should I chose?

Two surprising situations

* Imagine that the free throw percentage of player A has been better than the percentage of player B in every quarter of a given match. Can you be sure that the total percentage of player A during the match was better?

	1^{er} C	2º C	3^{er} C	4º C	Total
A	100%	100%	100%	60%	?
B	80%	80%	90%	50%	?

	1^{er} C	2º C	3^{er} C	4º C	Total
A	$1/1$	$2/2$	$1/1$	$12/20$	$\boxed{16/24 = 66,6\%}$
B	$8/10$	$16/20$	$9/10$	$1/2$	$\boxed{34/42 = 80,1\%}$

* This phenomenon is known as Simpson's paradox.

* A video explaining the paradox (in Spanish)

Percentages always add up to 100?

* The owner of a coffee shop studies the money he made from several products at the end of the week. He discovers that he lost 500 euros selling natural juices, and he won 700 euros selling cakes and 800 selling coffee.
  1. what percentage of the total profit comes from selling cakes?
  2. what percentage of the total profit comes from selling coffee?

* This type of phenomenon is common, for instance, when you read news about unemployment variations.

Problems

* An example to understand the soda problem (http://blog.mrmeyer.com/2011/wcydwt-coke-v-sprite/)

  You have two boxes, one with 100 blue chips and the other one with 100 red chips. You take 25 blue chips from the first box and put them in the second box. Now, you mix the chips and take 25 chips (in the right proportion) back to the first box. In which box are there more chips with the original color?

* We know that in a party there are 300 guests, and 99% of them are woman. How many woman should leave the party if we want that the percentage decreases to 98%?