Equipartitioning triangles

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Abstract

An intriguing conjecture of Nandakumar and Ramana Rao is that for every convex body $K \subseteq R^2$, and for any positive integer n, K can be expressed as the union of n convex sets with disjoint interiors and each having the same area and perimeter. The first difficult case - $n = 3$ - was settled by Bárány, Blagojević, and Szucs using powerful tools from algebra and equivariant topology. Here we give an elementary proof of this result in case K is a triangle.

Introduction

Let K be a convex body in the plane. Nandakumar and Ramana Rao [6] noticed that if a ham-sandwich cut for K were rotated through π radians - always maintaining a bisection of K - then at some point in this process, K is partitioned into two convex parts with disjoint interiors, and each having the same area and perimeter. A slightly more careful argument using this fact, along with induction, was given to show that for $n = 2^k$, K can always be partitioned into n convex subsets, each with the same area and perimeter. They made the intriguing

Conjecture 1 *For every $n \in N$ and all convex bodies $K \subseteq R^2$, K is the disjoint union of n convex pieces, each with the same area and perimeter.*

The conjecture describes an $n-$*equipartition* of K (because of the n equal areas) which is in addition *fair*, by virtue of the equal perimeters.

Bárány et. al. [2], using heavy-duty tools from algebra and equivariant topology settled the case $n = 3$: A $3-$*fan* is a point P in the plane with three rays emanating from it. It is convex if all angles are at most π. It equipartitions K if the three rays divide K into three regions of equal area, and it is fair, if these regions also have equal perimeter. Bárány et. al. showed that there is a convex $3-$fan that makes a *fair equi-partition* of K. Subsequently, Aronov and Hubard [1] and then Karasev [5], showed that the conjecture was true for $n = p^k$, a prime power, and also in dimension $d \geq 2$, with "area" replaced by "volume" and "perimeter", by "surface area". Blagojević and Ziegler found some problems with the proofs in these two papers, so they established the results - and more - using different tools. In the present paper, in an attempt to understand some of the geometric features of this problem and why - or why not - it may be difficult, we use (only) elementary methods to study the conjecture for R^2, and when K is a triangle. We call a $3-$fan *interior for* K if the apex P is interior to K; otherwise it is *exterior*. In the first case, all three rays play a role in the partition. In the second case, the partition is simply via two chords (which might meet on the boundary of K, but not in its interior). The first fact is

Theorem 1 *Every triangle has a fair, equi-partitioning, exterior $3-$fan.*

We are able to understand this problem in the triangle case partly due to a simple tool that describes how perimeters change when a chord in a partition is moved slightly while still preserving the areas of all regions.

Let AB and $|AB|$ be, respectively, the segment defined by points A and B and its length. Vector $\vec{OP}$ and point P will be identified if the context is clear enough. The list of points $AB \cdots D$ will be used to denote the corresponding polygon and, finally, $\pi(\cdot)$ will be the perimeter of the polygon.

Lemma 2 *Consider points $P \in OB$ and $Q \in OC$ such that $|AP| \leq |AQ|$. Let $P' = tP$ and let $Q' = \frac{1}{t}Q$ (then the area of OPQ equals the area of $OP'Q'$).*

1. *$\pi(OP'Q')$ and $\pi(BCQ'P')$ are convex functions of t, achieving the minimum when $|OP'| = |OQ'|$.*

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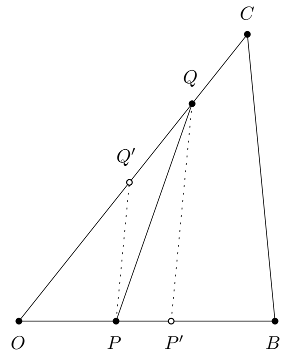
2. *If we write* $\Delta\pi_1 = \pi(AP'Q') - \pi(APQ)$ *and* $\Delta\pi_2 = \pi(BCQ'P') - \pi(BCQ'P')$ *then* $|\Delta\pi_1| \geq |\Delta\pi_2|$.



In the next section we describe some ideas behind the proofs for our results. The full proofs will appear in the actual paper.

1 Some Details

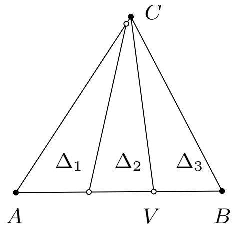
We fix a triangle $\Delta = ABC$, with $A = (0,0), B = (b,0)$, and $C = (x,y)$, $b, x, y \geq 0$, as in the figure above. Without loss of generality we put the smallest angle at C (so all sides have length at least b). If we take points $U = (b/3, 0)$ and $V = (2b/3, 0)$, Δ is partitioned into $\Delta_1 = CAU, \Delta_2 = CUV$, and $\Delta_3 = CVB$, *all with the same area*; (we note that this partition is by a 3−fan with apex at C, and having one ray is outside Δ). The goal is to rotate the chords CU and CV in such a way as to maintain equality of all three areas, but in such a way as to force all three perimeters to coincide. To unambiguously describe this process, we place points D and E on the boundary of Δ. Initially both points are placed on C. We then manipulate the chords DU and EV by moving the endpoints along the boundary of Δ, thus altering the three sets in the partition. We use the notation π_i to denote the perimeter of region i.

The case where $x = b/2$, and Δ is isosceles, is easiest. Here, we move chord DU counter-clockwise (maintaining the area of $\Delta_1 = DAU$), and chord EV clockwise (maintaining the area of $\Delta_3 = EVB$) until U and V coincide at $F = (b/2, 0)$. During this process the middle region is a pentagon $\Delta_2 = CDUVE$, ending at $\Delta_2 = CDFE$. If we also keep $U + V = (b, 0)$, and both U and V the same distance from $(b/2, 0)$, there will be a position where the partition is fair, by the intermediate value theorem, since Δ_1 and Δ_3 initially have equal perimeters, but larger than that of Δ_2, and at the end, $\pi_1 = \pi_3 \leq \pi_2$, by virtue of AB being the smallest side of Δ.

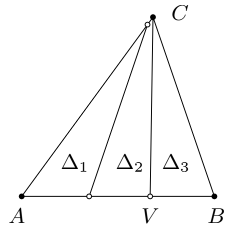
When Δ is not isosceles, we can take $x \geq b/2$ (the other case being symmetric), so C is to the right of the midpoint of AB and π_1 is larger than either π_2 or π_3. There are two different possibilities when we begin moving the chords: either $x \geq 2b/3$ and we start with $\pi_1 \geq \pi_2 \geq \pi_3$, or $x \leq 2b/3$ and we start with $\pi_1 \geq \pi_3 \geq \pi_2$.

For the second, we use the same approach as in the isosceles case, moving DU counterclockwise, and EV clockwise - always preserving equal areas. First, DU moves alone until the perimeters of Δ_1 and Δ_3 are the same: Lemma 1 shows that both π_1 and π_2 decrease, but π_1 more quickly, so at some point, we will have $\pi_1 = \pi_3 > \pi_2$ (see case 1 in the figure, below). At this point EV also begins to move, but clockwise so as to preserve three equal areas, but also preserving $\pi_1 = \pi_3$. This occurs because, by Lemma 1, as EV rotates clockwise - preserving areas - π_3 and π_2 both decrease. The process ends when U and V coincide and just as in the isosceles case, the intermediate value theorem assures that there is a position in this process where we have a fair 3−equipartition.

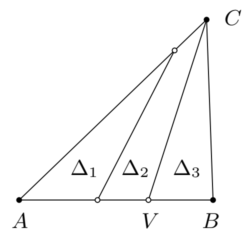
When $x > 2b/3$ the plan is to move both DU and EV, always preserving equal areas for the triangles of the partition. First DU moves alone until a point is reached where two of the perimeters are the same. Lemma 1 shows that both π_1 and π_2 decrease, the first more rapidly. Depending on the location of C, we will have either $\pi_1 > \pi_2 = \pi_3$ (see case 2 in the Figure 1) or $\pi_1 = \pi_2 > \pi_3$ (see case 3). In case 3 we now also move EV counterclockwise so as to preserve the area of Δ_3 and, according to Lemma 1, increase π_3 while reducing π_2. The net result is that the *coordinated* counterclockwise rotations of DU and EV can maintain the equality of π_1 and π_2 while at the same time reduce the difference between π_2 and $\pi_3 < \pi_2$. This process terminates either when DAU is isosceles or EV is parallel to CB and in both cases π_3 is the largest perimeter; this means we passed a position where all perimeters were equal. The argument for case 2 has EV rotating clockwise, and is similar to case 1, so we omit it here.



Case 1: $\pi_1 = \pi_3 > \pi_2$



Case 2: $\pi_2 = \pi_3 < \pi_1$



Case 3: $\pi_1 = \pi_2 > \pi_3$

The approach of the arguments above can be applied to prove the conjecture when the convex body is an isosceles triangle. Take points $V_i = (ib/n, 0)$ and points $U_i, i = 0, \ldots, n$; initially all $U_i = C$. The $n-1$ chords $U_iV_i, i = 1, \ldots, n-1$ partition Δ into n triangles ($\Delta_i = U_{i-1}V_{i-1}V_i, i = 1, \ldots, n$), of equal area. In general, chords $U_iV_i, i \leq n/2$ will be rotated counter clockwise, and $U_iV_i, i \geq n/2$ rotated clockwise, so as to preserve areas. First, we rotate U_1V_1 counterclockwise and $U_{n-1}V_{n-1}$ clockwise by the same amount, but preserving the areas of Δ_1 and Δ_{n-1}. Lemma 1 and the intermediate value theorem imply that there is a point where perimeters $P_1 = P_2 = P_{n-2} = P_{n-1}$. Now we move both U_1V_1 and U_2V_2 counterclockwise in a *coordinated* way so as to preserve equal areas *and perimeters* of Δ_1 and Δ_2, and also make the corresponding clockwise rotation of $U_{n-2}V_{n-2}$ and $U_{n-1}V_{n-1}$ to maintain the equal areas and perimeters for Δ_{n-2} and Δ_{n-1}. We deduce there is a point where we will have *six* triangles with equal areas and perimeters, etc.

2 Discussion

It is clear that an equilateral triangle has two distinct fair 3−equipartitions that are interior (as well as three exterior ones - for each vertex, the process outlined in the previous section produces an equi-partition with an exterior 3-fan), and that a rectangle has both interior and exterior ones. Also it is clear that a circle only has interior 3−fans. It would be interesting to understand when - assuming K has a fair, equi-partitioning 3−fan - whether it may be interior or exterior, or perhaps either.

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