# New results on lower bounds for the number of ($\leq k$)-facets *

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**Abstract**

In this paper we present two different results dealing with the number of ($\leq k$)-facets of a set of points:

1. We give structural properties of sets in the plane that achieve the optimal lower bound $3\binom{k+2}{2}$ of ($\leq k$)-edges for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$;
2. We show that for $k < \lfloor n/(d+1) \rfloor$ the number of ($\leq k$)-facets of a set of $n$ points in general position in $\mathbb{R}^d$ is at least $(d+1)\binom{k+d}{d}$, and that this bound is tight in the given range of $k$.

## 1 Introduction

In this paper we deal with the problem of giving lower bounds to the number of ($\leq k$)-facets of a set of points $S$: An oriented simplex with vertices at points of $S$ is said to be a *$k$-facet* of $S$ if it has exactly $k$ points in the positive side of its affine hull. Similarly, the simplex is said to be an *($\leq k$)-facet* if it has at most $k$ points in the positive side of its affine hull. If $S \subset \mathbb{R}^2$, a $k$-facet of $S$ is usually named a *$k$-edge*.

The number of $k$-facets of $S$ is denoted by $e_k(S)$, and $E_k(S) = \sum_{j=0}^{k} e_j(S)$ is the number of ($\leq k$)-facets (the set $S$ will be omitted when it is clear from the context). Giving bounds on these quantities, and on the number of the companion concept of $k$-sets, is one of the central problems in Discrete and Computational Geometry, and has a

*Part of the research on this paper was carried out while the first author was a visiting professor at the Mathematics Department, University of Alcalá, Spain.

†Research partially supported by the FWF (Austrian Fonds zur Förderung der Wissenschaftlichen Forschung) under grant S09205, NFN Industrial Geometry.

‡Research partially supported by grants MCYT TIC2002-01541, and HU2007-0017.

§Research partially supported by grants MTM2005-08618-C02-02, S-0505/DPI/0235-02, HU2007-0017, and MTM2008-04699-C03-02.

¶Research partially supported by grants TIC2003-08933-C02-01, S-0505/DPI/0235-02, HU2007-0017, and MTM2008-04699-C03-02.

long history that we will not try to summarize here. Chapter 8.3 in [5] is a complete and up to date survey of results and open problems in the area.

Regarding lower bounds for $E_k(S)$, which is the main topic of this paper, the problem was first studied by Edelsbrunner et al. [7] due to its connections with the complexity of higher order Voronoi diagrams. In that paper it was stated that, in $\mathbb{R}^2$,

$$E_k(S) \geq 3\binom{k+2}{2} \tag{1}$$

and it was given an example showing tightness for $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The proof used circular sequences but, unfortunately, contained an unpluggable gap, as pointed out by Lovász et al. [9]. A correct proof, also using circular sequences, was independently found by Ábrego and Fernández-Merchant [1] and Lovász et al. [9]. In both papers a strong connection was discovered between the number of ($\leq k$)-edges and the number of convex quadrilaterals in a point set $S$. Specifically, if $\square(S)$ denotes the number of convex quadrilaterals in $S$, in [9] it was shown that

$$\square(S) = \sum_{k<\frac{n-2}{2}} (n-2k-3)\, E_k(S) - \frac{3}{4}\binom{n}{3} + c_n, \tag{2}$$

where

$$c_n = \begin{cases} \frac{1}{4} E_{\frac{n-3}{2}}(S), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Giving lower bounds for $\square(S)$ is in turn equivalent to determining the rectilinear crossing number of the complete graph: if we draw $K_n$ on top of a set of points $S$, then the number of intersections in the drawing is exactly the number of convex quadrilaterals in $S$. The interested reader can go through the extensive online bibliography by Vrt'o [10] where the focus is on the problem of crossing numbers of graphs.

The lower bound in Equation 1 was slightly improved for $k \geq \lfloor \frac{n}{3} \rfloor$ by Balogh and Salazar [4], again using circular sequences. Using different techniques, and based on the observation that it suffices to proof the bound for sets with triangular convex hull, we have recently shown [2] that, in $\mathbb{R}^2$,

$$E_k(S) \geq 3\binom{k+2}{2} + \sum_{j=\lfloor \frac{n}{3} \rfloor}^{k} (3j - n + 3). \tag{3}$$

If $n$ is divisible by 3, this expression can be written as

$$E_k(S) \geq 3\binom{k+2}{2} + 3\binom{k - \frac{n}{3} + 2}{2}.$$

In this paper we deal with two different problems related to lower bounds for $E_k$: In Section 2, we study structural properties of those sets in $\mathbb{R}^2$ that achieve the lower bound in Equation 1 for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The main result of this section is that if $E_k(S)$ is minimum for a given $k$, then $E_j(S)$ is also minimum for every $0 \leq j < k$. In Section 3 we study the $d$-dimensional version of the problem and show that, for a set of $n$ points in general position in $\mathbb{R}^d$,

$$E_k(S) \geq (d+1)\binom{k+d}{d}, \text{ for } 0 \leq k < \lfloor \tfrac{n}{d+1} \rfloor, \tag{4}$$

and that this bound is tight in that range. To the best of our knowledge, this is the first result of this kind in $\mathbb{R}^d$.

## 2 Optimal sets for $(\leq k)$-edge vectors

Given $S \subset \mathbb{R}^2$, let us denote by $\mathcal{E}_k(S)$ the set of all $(\leq k)$-edges of $S$, hence $E_k(S)$ is the cardinality of $\mathcal{E}_k(S)$. Throughout this section we consider $k \leq \lfloor \frac{n}{3} \rfloor - 1$. Recall that for a fixed such $k$, $E_k(S)$ is *optimal* if $E_k(S) = 3\binom{k+2}{2}$. Recall also that, by definition, a $j$-edge has exactly $j$ points of $S$ in the positive side of its affine hull, which in this case is the open half-plane to the right of its supporting line.

We start by giving a new, simple, and self-contained proof of the bound in Equation 1, using a new technique which will be useful in the rest of the section. Although in this section they will be used in $\mathbb{R}^2$, the following notions are presented in $\mathbb{R}^d$ for the sake of generality and in view of Section 3.

**Definition 1** ([8])**.** Let $S$ be a set of $n$ points and $\mathcal{H}$ a family of sets in $\mathbb{R}^d$. A subset $N \subset S$ is called an *$\epsilon$-net* of $S$ (with respect to $\mathcal{H}$) if for every $H \in \mathcal{H}$ such that $|H \cap S| > \epsilon n$ we have that $H \cap N \neq \varnothing$.

**Definition 2.** A *simplicial $\epsilon$-net* of $S \subset \mathbb{R}^d$ is a set of $d+1$ vertices of the convex hull of $S$ that are an $\epsilon$-net of $S$ with respect to closed half-spaces. A simplicial $\frac{1}{2}$-net will be called a *simplicial half-net.*

**Lemma 3.** *Every set $S \subset \mathbb{R}^2$ of $n$ points has a simplicial half-net.*

*Proof.* Let $T$ be a triangle spanned by three vertices of the convex hull of $S$. An edge $e$ of $T$ is called *good* if the closed half-plane of its supporting line which contains the third vertex of $T$, contains at least $\frac{n}{2}$ points from $S$. $T$ is called *good* if it consists of three good edges. Clearly, the vertices of a good triangle $T$ are a simplicial half-net of $S$; the vertices of $T$ being of the convex hull implies that the intersection of $S$ with a half-plane not containing any vertex of $T$ lies in the complement of some of the three half-planes defined by the good edges.

Let $T$ be an arbitrary triangle spanned by vertices of the convex hull of $S$ and assume that $T$ is not good. Then observe that only one edge $e$ of $T$ is not good and let $v$ be the vertex of $T$ not incident to $e$. Choose a point $v'$ of the convex hull of $S$ opposite to $v$ with respect to $e$. Then $e$ and $v'$ induce a triangle $T'$ in which $e$ is a good edge. If $T'$ is a good triangle we are done. Otherwise we iterate this process. As the cardinalities of the subsets of vertices of $S$ considered are strictly decreasing (the subsets being restricted by the half-plane induced by $e$), the process terminates with a good triangle. □

**Theorem 4.** *For every set $S$ of $n$ points and $0 \leq k < \lfloor \frac{n-2}{2} \rfloor$ we have $E_k(S) \geq 3\binom{k+2}{2}$.*

*Proof.* The proof goes by induction on $n$. From Lemma 3, we can guarantee the existence of $T = \{a, b, c\} \subset S$, an $\frac{1}{2}$-net made up with vertices of the convex hull.

Let $S' = S \setminus T$ and consider an edge $e \in \mathcal{E}_{k-2}(S')$. We observe that $T$ cannot be to the right of $e$: there are at least $\frac{n}{2}$ points on the closed half-plane to the left of $e$ and that would contradict the definition of $\frac{1}{2}$-net. Therefore, $e \in \mathcal{E}_k(S)$.

If we denote by $\mathcal{ET}_k(S)$ the set of $(\leq k)$-edges of $S$ incident to points in $T$, we have that

$$\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) \subset \mathcal{E}_k(S). \tag{5}$$

There are $2(k+1)$ $(\leq k)$-edges incident to each of the convex hull vertices $a, b, c$ (which can be obtained rotating a ray based on that vertex). We observe that at most three edges of $\mathcal{ET}_k(S)$ might be incident to two points of $T$ (those of the triangle $T$) and that the union in Equation 5 is disjoint. Therefore, using the induction hypothesis we have

$$E_k(S) \geq E_{k-2}(S') + 3 + 6k \geq 3\binom{k}{2} + 3 + 6k = 3\binom{k+2}{2}. \tag{6}$$

□

**Corollary 5.** *Let $S$ be a set of $n$ points, $T = \{a, b, c\}$ a simplicial half-net of $S$ and $S' = S \setminus T$. If $E_k(S) = 3\binom{k+2}{2}$ then:*

*(a)* $E_{k-2}(S') = 3\binom{k}{2}$.

*(b) A $k$-edge of $S$ is either a $(k-2)$-edge of $S'$ or is incident to a point in $T$.*

*Proof.* If $E_k(S) = 3\binom{k+2}{2}$, both inequalities in Equation 6 are tight. Therefore $E_{k-2}(S') = 3\binom{k}{2}$ and Equation 5 becomes $\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) = \mathcal{E}_k(S)$ (disjoint union) which trivially implies part (b). □

**Theorem 6.** *If $E_k(S) = 3\binom{k+2}{2}$, then $S$ has a triangular convex hull.*

*Proof.* We prove the statement by induction over $k$. For $k = 0$ nothing has to be proven, so let $k = 1$, assume that $E_1 = 9$ and let $h = |CH(S)|$. We have $h$ 0-edges and at least $h$ 1-edges (two per convex hull vertex, but each edge might be counted twice). Thus $E_1 = 9 \geq 2h$ and therefore $h \leq 4$. Assume now $h = 4$. Then at most two 1-edges can be counted twice, namely the two diagonals of the convex hull. Thus we have $4 + 8 - 2 = 10$ $(\leq 1)$-edges and we conclude that if $E_1 = 9$, then $S$ has a triangular convex hull.

For the general case consider $k \geq 2$, let $T = \{a, b, c\}$ be the simplicial half-net guaranteed by Lemma 3 and let $S' = S \setminus T$. From Corollary 5, part (a), we know that $E_{k-2}(S') = 3\binom{k}{2}$ and, by induction, we may assume that $S'$ has a triangular convex hull. Moreover, from part (b), no $(k-1)$-edge of $S'$ can be an $(\leq k)$-edge of $S$ and, therefore, any $(k-1)$-edge of $S'$ must have two vertices of $T$ on its positive side. Consider the six $(k-1)$-edges of $S'$ incident to the three convex hull vertices of $S'$: See Figure 1, where the supporting lines of these $(k-1)$-edges are drawn as dashed lines and $S'$ is depicted as the central triangle. Each cell outside $S'$ in the arrangement of the supporting lines contains a number counting the $(k-1)$-edges considered which have that cell on their positive side. A simple counting argument shows that the only way of placing the three vertices $a, b, c$ of $T$ such that each $(k-1)$-edge of $S'$ drawn has two of them on its positive side is to place one in each cell labeled with a 4. We conclude that no vertex of $S'$ can be on the convex hull of $S$ and the theorem follows. □

**Corollary 7.** *If $E_k(S) = 3\binom{k+2}{2}$, then the outermost $\lceil \frac{k}{2} \rceil$ layers of $S$ are triangles.*

*Proof.* From the optimality for $E_k(S)$ and using the same argument as in the proof of Theorem 6, it follows that we can iteratively remove the outermost $\lceil \frac{k}{2} \rceil$ layers to obtain optimal subsets, which, by Theorem 6, have triangular convex hulls. □

**Theorem 8.** *If $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S) = 3\binom{j+2}{2}$ for every $0 \leq j \leq k$.*

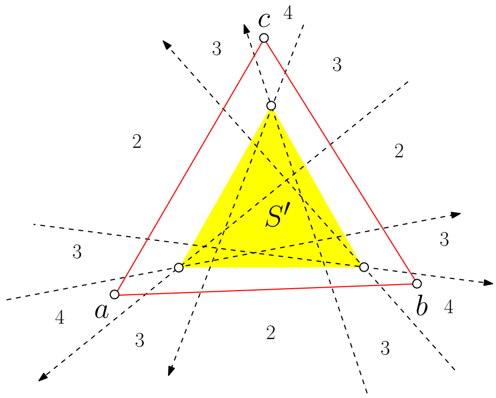


Figure 1: Each $(k-1)$-edge of $S'$ incident to a convex hull vertex of $S'$ (supporting lines are shown as dashed lines) has two vertices of $T$ on its positive side.

*Proof.* We prove the theorem by induction on $k$. For $k = 0, 1$ the theorem is equivalent to Theorem 6, so let $k \geq 2$. It is sufficient to show that optimality of $E_k(S)$ implies optimality of $E_{k-1}(S)$, as the present theorem follows by induction.

Let $T$ be the vertices of $CH(S)$ (which is a triangle as guaranteed by Theorem 6) and let $S' = S \setminus T$. As in Theorem 4 we have

$$\mathcal{E}_{k-3}(S') \cup \mathcal{ET}_{k-1}(S) \subset \mathcal{E}_{k-1}(S).$$

Observe that $E_{k-2}(S')$ is optimal, as guaranteed by Corollary 5 and this implies optimality of $E_{k-3}(S')$ by induction. $|\mathcal{ET}_{k-1}(S)|$ is also optimal because the convex hull of $S$ is the triangle $T$. Therefore, to prove optimality of $E_{k-1}(S)$ it only remains to show that no $(k-2)$-edge of $S'$ can be a $(k-1)$-edge of $S$.

So let $e$ be a $(k-2)$-edge of $S'$ and let $p$ and $q$ be the vertices of the convex hull of $S'$ incident to $e$ or on its positive side. The existence of $p$ and $q$ is guaranteed by Corollary 5, part (b). Without loss of generality, assume that the edge $pq$ is horizontal with the remaining vertices of $S'$ above it, see Figure 2 for the rest of the proof. Let $\ell_1$ be the $(k-1)$-edge of $S'$ incident to $p$ which has $q$ on its positive side and $\ell_2$ the $(k-1)$-edge incident to $q$ and having $p$ on its positive side. The *boundary chain* is the lower envelope of $\ell_1$, $pq$, and $\ell_2$. We claim that $e$ does not intersect the boundary chain and lies above it. If $e$ is incident to $p$ or $q$ then the claim is obviously true. Otherwise observe that $e$ has to intersect the supporting lines of both considered $(k-1)$-edges in the interior of $S'$, as otherwise there would be too many vertices on the positive side of $e$. But then again $e$ lies above the boundary chain and the claim follows.

From the proof of Theorem 6 we know that two of the vertices of the convex hull of $S$ have to lie below our boundary chain (below the $(k-1)$-edges, see $a$ and $b$ in Figure 2) and thus on the positive side of $e$. Therefore $e$ has at least $k$ vertices of $S$ on its positive side and does not belong to $\mathcal{E}_{k-1}(S)$. We conclude that $E_{k-1}(S)$ is optimal and the theorem follows. $\square$

**Corollary 9.** *Let $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$. If $E_k(S) = 3\binom{k+2}{2}$, then $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$.*

## 3 A lower bound for $(\leq k)$-facets in $\mathbb{R}^d$

Throughout this section, $S \subset \mathbb{R}^d$ will be a set of $n$ points in general position.

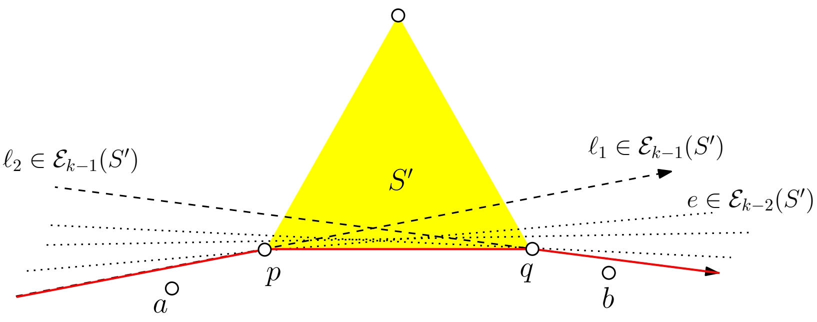


Figure 2: All $(k-2)$-edges of $S'$ (supporting lines are shown as dotted lines) lie above the (bold) lower envelope.

We remind that $e_k(S)$ and $E_k(S)$ denote, respectively, the number of $k$-facets and the number of $(\leq k)$-facets of $S$. The main result of this section is a lower bound for the number of $(\leq k)$-facets of a set of $n$ points in general position in $\mathbb{R}^d$ in the range $0 \leq k < \lfloor \frac{n}{d+1} \rfloor$.

The proof follows the approach in Theorem 4, using the fact that every set of points has a centerpoint: a point $c \in \mathbb{R}^d$ is a *centerpoint* of $S$ if no open half-space that avoids $c$ contains more than $\lceil \frac{dn}{d+1} \rceil$ points of $S$ (see [6]).

**Theorem 10.** *Let $S$ be a set of $n \geq d+1$ points in $\mathbb{R}^d$ in general position. Then*

$$E_k(S) \geq (d+1)\binom{k+d}{d} \qquad \text{if } 0 \leq k < \lfloor \tfrac{n}{d+1} \rfloor.$$

*Furthermore, the bound on $E_k(S)$ is tight in the given range of $k$.*

*Proof.* The proof uses induction on $n$ and $d$. The base case for $n = d+1$ is obvious and for $d = 2$ is just Equation 1.

Let $k < \lfloor \frac{n}{d+1} \rfloor$ and let $c$ be a centerpoint of $S$. Let us consider a simplex $T$ containing $c$ with vertices in the convex hull of $S$, and let $S' = S \setminus T$. From the definition of centerpoint, it follows that no open half-space that avoids $T$ contains more than $\lceil \frac{dn}{d+1} \rceil - 1$ points or, equivalently, every closed half-space containing $T$ has at least $\lfloor \frac{n}{d+1} \rfloor + 1$ points.

We denote by $\mathcal{E}_k^j(S)$ the set of $(\leq k)$-facets of $S$ incident to exactly $j$ vertices of $T$, and $E_k^j(S)$ will be the cardinality of $\mathcal{E}_k^j(S)$.

For $j = 0$, we observe that $\mathcal{E}_{k-d}^0(S') \subset \mathcal{E}_k^0(S)$, because a closed half-space containing at most $k$ points cannot contain all the vertices of $T$. Because $k - d \leq \lfloor \frac{n-(d+1)}{d+1} \rfloor - 1$, we can apply induction on $n$ and get

$$E_k^0(S) \geq E_{k-d}^0(S') \geq (d+1)\binom{(k-d)+d}{d} = (d+1)\binom{k}{d}.$$

For $1 \leq j \leq d$, let $T_j$ be a subset of $j$ vertices of $T$ and let $S_\pi$ be the projection from $T_j$ of $S \setminus T$ onto the $(d-j)$-dimensional subspace $\pi$ defined by the points in $T \setminus T_j$: a point $p \in S \setminus T$ is mapped to the intersection between the $j$-flat defined by $p$ and $T_j$ and the $(d-j)$-flat defined by points in $T \setminus T_j$. Using the general position assumption, it is easy to see that the intersection has dimension zero. If the intersection were empty, we could slightly perturb $p$ without changing the number of $(\leq k)$-facets of $S$.

Now, if $\sigma \subset S_\pi$ is an $(\leq (k-d+j))$-facet of $S_\pi$, then $\sigma \cup T_j$ is an $(\leq k)$-facet of $S$; clearly $\sigma \cup T_j$ is an $(\leq k+1)$-facet, and it cannot be a $(k+1)$-facet because there would be a closed half-space containing $c$ and only $k+1 \leq \lceil \frac{n}{d+1} \rceil$ points of $S$. Because

$$k-d+j \leq \left\lfloor \frac{n}{d+1} \right\rfloor - 1 \leq \left\lfloor \frac{n-j}{d-j+1} \right\rfloor - 1$$

we can apply induction on $d$ and $n$, obtaining that there are at least

$$(d-j+1)\binom{k-d+j+(d-j)}{d-j} = (d-j+1)\binom{k}{d-j}$$

$(\leq k)$-facets of $S$ incident to $T_j$. Summing over all the subsets of $j$ points of $T$, we get

$$E_k^j(S) \geq \binom{d+1}{j}(d-j+1)\binom{k}{d-j},$$

and, finally,

$$E_k(S) \geq \sum_{j=0}^{d} \binom{d+1}{j}(d-j+1)\binom{k}{d-j} = (d+1)\binom{k+d}{d}.$$

As for tightness, the example showing that the bound $3\binom{k+2}{2}$ is tight for $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$ in the planar case [7] can be extended to $\mathbb{R}^d$: Consider $d+1$ rays in $\mathbb{R}^d$ emanating from the origin and with the property that any hyperplane containing one of them leaves on each open half-space at least one of the remaining rays. For instance, we could take the rays defined by the origin and the vertices of a regular simplex inscribed in the unit $d$-sphere.

Let $n = (d+1)m$ and put chains $C_1, \ldots, C_{d+1}$ with $m$ points on each ray, slightly perturbed to achieve general position. For $j < m$, every $j$-facet of $S$ is defined by $d$ points on different chains, because a facet defined by two points in the same chain has at least $m$ points on each half-space. If we label the points of each chain from 0 to $m-1$ (starting from the convex hull) and consider $p_{i_1}^1 \in C_1, \ldots, p_{i_d}^d \in C_d$, they define a $(i_1 + \ldots + i_d)$-facet. Therefore, the number of $(\leq k)$-facets defined by one point on each of these chains equals the cardinality of the set

$$\{(i_1, \ldots, i_d) \in \mathbb{Z}^d : i_1 + \ldots + i_d \leq k,\ 0 \leq i_1, \ldots, i_d \leq k\},$$

which is exactly $\binom{k+d}{d}$. Since these are the facets defined by points in $d$ out of the $d+1$ chains, the total number of $(\leq k)$-facets of the set is exactly $(d+1)\binom{k+d}{d}$. □

# 4 Conclusions and open problems

For $S \subset \mathbb{R}^2$ we have shown that, for a fixed $k \leq \lfloor \frac{n}{3} \rfloor - 1$, if $E_k(S)$ is optimal, i.e. $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S)$ is also optimal in the whole range $0 \leq j \leq k$, which in turn implies that $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$. Moreover, then the outermost $\lceil \frac{k}{2} \rceil$ layers of $S$ are triangles and these layers consist entirely of $j$-edges of special types. In addition, we have been able to give a simple construction showing that the lower bound in Equation 3 is tight for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$.

All these results reveal significant deeper insight into the structure of sets minimizing the number of $k$-edges, the final goal being to find tight bounds for every $k$.

Moreover, for an $n$-point set $S \subset \mathbb{R}^d$ we have proven the lower bound $(d+1)\binom{k+d}{d}$ for the number of $(\leq k)$-facets in the range $0 \leq k < \lfloor n/(d+1) \rfloor$, which is the first result of this kind in $\mathbb{R}^d$.

The restriction $k < \lfloor n/(d+1) \rfloor$ stems from the underlying technique, namely using the centerpoint of a set, and can probably be removed. An alternative proof of Theorem 10, using a simplicial half-net instead of a centerpoint, would be sufficient to extend the bound to the whole range of $k$. Therefore, it is a challenging task to extend Lemma 3 to dimension $d$, as the following conjecture states:

**Conjecture 11.** *Every point set $S \subset \mathbb{R}^d$ has a simplicial half-net.*

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