On Structural and Graph Theoretic Properties of Higher Order Delaunay Graphs*

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Abstract

Given a set P of n points in the plane, the order-k Delaunay graph is a graph with vertex set P and an edge exists between two points $p, q \in P$ when there is a circle through p and q with at most k other points of P in its interior. We provide upper and lower bounds on the number of edges in an order-k Delaunay graph. We study the combinatorial structure of the set of triangulations that can be constructed with edges of this graph. Furthermore, we show that the order-k Delaunay graph is connected under the flip operation when $k \leq 1$ but not necessarily connected for other values of k. If P is in convex position then the order-k Delaunay graph is connected for all $k \geq 0$. We show that the order-k Gabriel graph, a subgraph of the order-k Delaunay graph, is Hamiltonian for $k \geq 15$. Finally, the order-k Delaunay graph can be used to efficiently solve a coloring problem with applications to frequency assignments in cellular networks.

1 Introduction and preliminary definitions

Let P be a set of n points in the plane in general position (i.e., no three points are collinear). A geometric graph is a graph whose vertex set is P and whose edge set is a set of segments joining pairs of vertices. The *order-k Delaunay graph of P*, hereafter denoted as k-DG(P), is the geometric graph formed by the edges of P with order at most k. For two points $p, q \in P$, we say that the edge pq has *order k*, and write $o(pq) = k$, provided that every circle with p and q on its boundary contains at least k points of P in its interior

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and at least one of these circles contains exactly k points in its interior. For an edge pq of order k, a circle with p and q on its boundary containing exactly k points is called a *witness circle*. When $k = 0$, we obtain a relevant and ubiquitous structure, the Delaunay graph [6, 3, 13]. It is well known that this graph is a triangulation –usually denoted by $DT(P)$– when the points are in general position. In the remainder of the paper, all graphs are constructed on an arbitrary yet fixed point set P in general position. Therefore, we will write k-DG to mean k-DG(P), unless otherwise indicated. We include some comments in the concluding section on the case in which P contains degeneracies.

In this paper, we concentrate mainly on the graph theoretic properties of the order-k Delaunay graph as well as some applications arising from these properties. In particular, in Section 5, we show how the order-k Delaunay graph can be used to efficiently solve a coloring problem that can be applied to frequency assignments in cellular networks. The motivation for our study comes from three main directions –related to each other– that we describe next.

VORONOI DIAGRAM AND DELAUNAY GRAPH. The Voronoi diagram of a point set in the plane and its dual, the Delaunay graph, constitute one of the pillars of Computational Geometry theory and the main tool in the applications of the discipline. Surveys are given in [3, 13] and an encyclopedic treatment of these structures can be found in the book by Okabe et al. [26].

The order-k Voronoi diagram is a generalization that consists of the decomposition of the plane into regions that have the same set of k closest neighbors. The higher order Voronoi diagrams are related to the k-DG graphs. We outline the precise relation between these two structures in Section 3. Although from their earliest development these structures were used for several optimization purposes, it took some time to realize that their combinatorial properties should also be well understood. In particular, it was thought that Delaunay triangulations are Hamiltonian, but this was disproved by Dillencourt in [7]. However, he was able to prove in a subsequent work that $DT(P)$ is a 1-tough graph which implies that for even $|P|$ the graph $DT(P)$ contains a perfect matching [8]. A natural extension to this approach is to study whether higher order Delaunay graphs are Hamiltonian. In fact, Chang and Yang proved that this is always the case for 20-DG [5], but to the best of our knowledge, no better bounds have been obtained since then. One of the contributions presented here in Section 2 is that the order-k Gabriel graph, a subgraph of the order-k Delaunay graph, always contains a Hamiltonian cycle for $k \geq 15$. The proof of this result is rather technical and it remains elusive whether it holds for a much smaller k, as in fact we believe.

PROXIMITY GRAPHS. A proximity graph is a graph that represents the structure in a point set by joining pairs of points in the plane that satisfy some type of proximity measure. It is the measure that determines the type of graph that results. Many different measures of proximity have been defined and the resulting graphs have been studied both from a combinatorial and computational perspective [19]. For most of these graphs, an edge exists between a pair of points when some proximity region defined by this pair of points is empty. Proximity graphs find applications in many areas such as Pattern Recognition, Data Structures, Computational Geometry and Motion Planning, and there has been a recent flow of papers on their application to data depth analysis [1, 18, 27].

Among proximity graphs, the Delaunay graph/triangulation of a planar point set P is an especially relevant structure. Its edges have a simple geometric definition in terms of proximity measure. Two points $p, q \in P$ form a Delaunay edge provided that there exists a circle with p and q on its boundary with no points of P in its interior. Closely related to this is the Gabriel graph and the Relative Neighborhood graph. In the Gabriel graph, two points $p, q \in P$ form an edge when the circle with the segment pq as diameter has no points of P in its interior. An edge exists between two points $p, q \in P$ in the Relative Neighborhood graph if no $z \in P$ satisfies $\max\{d(p, z), d(q, z)\} < d(p, q)$. The containment relation between the proximity relations defining the Relative Neighborhood graph, the Gabriel graph and the Delaunay triangulation imply a containment relation between the graphs [26].

These proximity measures can be generalized in a natural way by relaxing the condition that the proximity regions need to be empty. The kth order Delaunay graph we are considering here is an example, in which two points $p, q \in P$ form an edge provided that there exists a circle with p and q on its boundary with at most k points of the set P inside the circle. Note again that the order-0 Delaunay graph is the standard one. This relaxation naturally leads to the definition of the order-k Gabriel graph, the order-k Relative Neighborhood graph and others.

These higher-order graphs have also been studied in the literature but not as extensively as their order-0 counterparts. In [30], properties of the order-k Gabriel Graph are investigated and an algorithm for its construction is proposed, while in [5] it is shown that the order-20 Relative Neighborhood graph is Hamiltonian. The size of these graphs is also a basic parameter that has attracted significant attention. In particular, it has been proved that the size of the kth order Relative Neighborhood graph is linear in kn [31] and that the size of the Sphere of Influence graph[1] is at most $15n$ [29].

For higher order Voronoi diagrams, the combinatorial complexity of these structures is related to the complexity of levels in arrangements of hyperplanes [2, 16]. These problems have a rich history, related to halving lines/hyperplanes (or more generally k-sets), and have been studied since the seminal works by Lovász, Erdős, and others. In Section 3, we provide upper and lower bounds on the number of edges in an order-k Delaunay graph, which is related to these topics.

TRIANGULATIONS. While the Delaunay triangulation optimizes several quality criteria among the triangulations of a point set, in some situations it may be preferrable to have some flexibility for modifying the structure to some extent, for example in the case of terrains, where points have elevations. Gudmundsson *et al.*[14] define the Delaunay order of a triangulation T as the maximum of the orders of the triangles in T, where the order of a triangle is defined as the number of points contained inside its circumscribing circle. They study the use of higher order Delaunay triangulations for optimization purposes. Variations for constrained triangulations, terrains and polygons have been con-

[1]The Open Sphere of Influence Graph is defined by considering a disk around each point with radius equal to the distance to that point's closest neighbour, and linking any two points with an edge if their disks intersect.

sidered in [15, 20, 28, 21]. In [14] Gudmundsson *et al.* also consider order-k Delaunay edges and study the problem of computing the set of order-k Delaunay edges which can be completed to a triangulation such that all the triangles have order at most k; however, order-k Delaunay graphs are not explicitly considered.

A complementary approach consists of considering the set of all triangulations whose edges are constrained to have order at most k and use the diagonal flip operation that since its introduction by Lawson has been used as a heuristic for several quality criteria [22, 4]. This raises the issue of the connectivity of the corresponding flip graph, and that is why in Section 4, we study the combinatorial structure of the set of triangulations that can be constructed with edges from k-DG and show that it is connected under the flip operation if $k \leq 1$ but not in general for other values, unless P is in convex position.

2 Order 15 Gabriel graph is Hamiltonian

In this section, we show that for all $k \geq 15$, the order-k Gabriel graph contains a Hamiltonian cycle. This shows that 15-DG is Hamiltonian, because it is known that the order k Gabriel graph (denoted k-GG) is a subgraph of the order k Delaunay graph. Although this result improves on the best previous result by Chang et al. [5], who showed that 20-DG is Hamiltonian, we think that it is still far from optimal, as we even conjecture that 1-DG is Hamiltonian.

The approach we take to prove this is the following. We first define a total order on all the Hamiltonian cycles through a given point set. We then show that the minimum element in this total order is a Hamiltonian cycle where every edge is contained in 15-GG. Before proving this theorem, we first need a useful geometric lemma about cones.

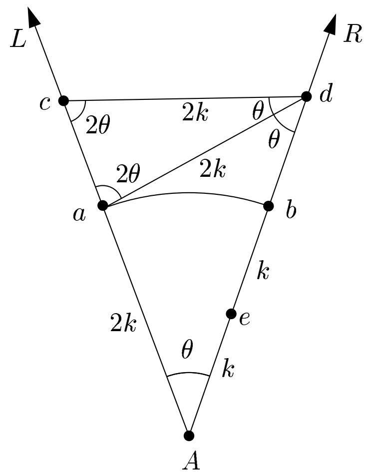

Figure 1: Illustration for Lemma 1

Lemma 1. *Let $0 < \theta \leq \pi/5$. Let $C(A, \theta, L, R)$ be a cone with apex A, bounding rays L and R emanating from A and angle θ computed clockwise from L to R. Given two points $x, y \in C(A, \theta, L, R)$ both of whose distance to A is greater than $2k$, for some constant $k > 0$, then either $d(x, y) < 2k$ or $d(x, y) < \max\{d(x, A) - k, d(y, A) - k\}$.*

Proof. First, consider the case where $\theta = \pi/5$. See Figure 1 for an illustration of cone $C(A, \theta, L, R)$. Let e be the point on R at distance k from A. The points c and d are on L and R respectively such that $|Ac| = |Ad|$ and $|\overline{cd}| = 2k$. The points a and b are on L and R, respectively, both at distance $2k$ from A. Since $\theta = \pi/5$ and triangles $\triangle(a, c, d)$ and $\triangle(A, c, d)$ are similar and isosceles, we have that $|\overline{ad}| = 2k$ and $d(A, d) = k/\sin(\pi/10) > 3.2k$.

Given two points $x, y \in C(A, \theta, L, R)$ we need to show that $d(x, y)$ has one of the two properties in the statement of the lemma. Without loss of generality, assume that x is further from A than y and that y is to the left of x (i.e., y is to the left of the line through A and x oriented from A to x). All other situations are symmetric.

If x is in the interior of the cone, rotate the cone counter-clockwise around A until x lies on R. Since y is to the left of x, the point y is still in the interior of the cone.

Since we assume that $2k < d(x, A)$, we have only two cases to consider: when $2k < d(x, A) \leq k/\sin(\pi/10)$ and $d(x, A) > k/\sin(\pi/10)$. In what follows, we use the notation $D(z, r)$ to represent a disk centered at point z with radius r.

Case 1: $(2k < d(x, A) \leq k/\sin(\pi/10))$ If $2k < d(x, A) \leq k/\sin(\pi/10)$, then both x and y are in the region defined by $C(A, \theta, L, R) \cap D(A, k/\sin(\pi/10)) \cap \overline{D(A, 2k)}$, where $\overline{D(A, 2k)}$ is the complement of the disk centered at A with radius $2k$. Since we noted above that the diameter of this region is $2k$, we have that $d(x, y) \leq 2k$. Now, the only time $d(x, y) = 2k$ is when x is on d and point y is either on a or c. However, when $x = d$, we have that $d(x, A) - k > d(x, y)$. Therefore, the conditions of the lemma have been met in this case.

Case 2: $(d(x, A) > k/\sin(\pi/10))$. When $d(x, A) > k/\sin(\pi/10)$ then by construction the furthest point from x is a, so we can assume that y is a. We need to show that $d(x, a) < d(x, A) - k$. This is equivalent to showing that $d(x, a) < d(x, e)$. Let $g = R \cap D(x, |\overline{ax}|)$. Consider the isosceles triangle $\triangle(g, a, x)$. Since $\angle(a, x, e) < \angle(a, d, e)$, the point g must be on the interior of the segment $\overline{eb}$. Therefore, $d(x, a) < d(x, A) - d(a, e)$ as required.

We have shown that the lemma holds for cones where $\theta = \pi/5$. Notice that the lemma trivially holds for cones whose angle is smaller since a cone of angle $\pi/5$ contains the cone with smaller angle. $\square$

The preceding lemma is used to show the following: if a minimum Hamiltonian cycle contains an edge e that is *not* 15-GG, then a set of edges (including e) can be deleted from the cycle followed by the insertion of a different set of edges to build a Hamiltonian cycle with lower weight, thereby contradicting the minimality of the original cycle.

Theorem 1. *Given a set P of n points in the plane in general position, the graph 15-GG contains a Hamiltonian cycle.*

Proof. Let H be the set of all Hamiltonian cycles through the points of P. Define a total order on the elements of H in the following way. Given an element $h \in H$, define the

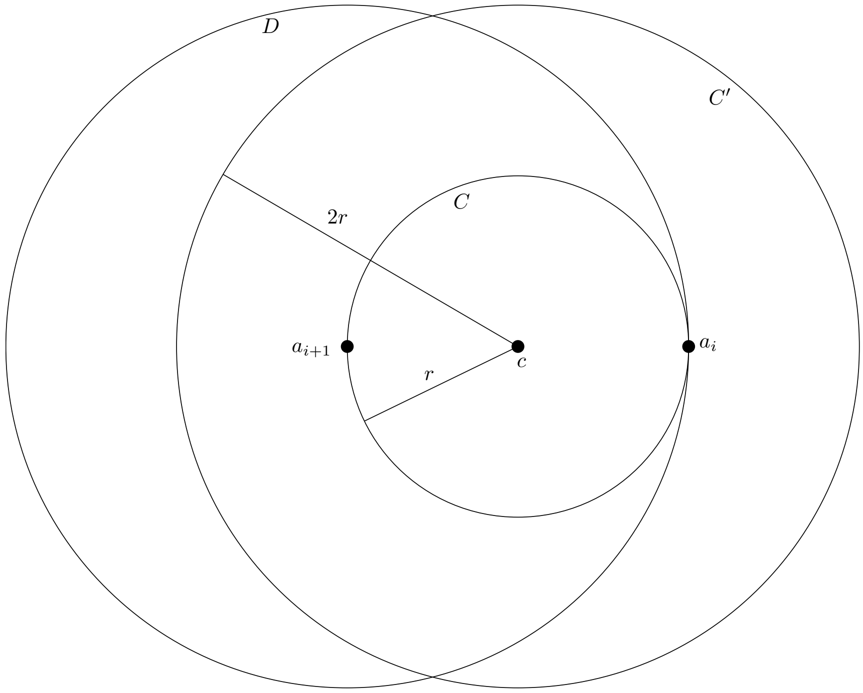

Figure 2: Illustration for Theorem 1

distance sequence of h as the lengths of the edges of the cycle sorted from longest to shortest, denoted $ds(h)$. Element $x \in H$ is greater than $y \in H$ if the first place where $ds(x)$ differs from $ds(y)$, the value in $ds(x)$ is greater than that of $ds(y)$. If two elements have exactly the same distance sequence, break ties arbitrarily to get a total order. Let $m = a_0, a_1, \ldots, a_{n-1}$ be the cycle in H with minimum distance sequence. We will show that all of the edges of m are in 15-GG. We proceed by contradiction.

Suppose that there are some edges in m that are not in 15-GG. Let $e = [a_i a_{i+1}]$ be the longest edge that is not in 15-GG (all index manipulation is modulo n). Let C be the circle with a_i and a_{i+1} as diameter. Let $r = |a_i a_{i+1}|/2$ be the radius of C.

Claim 1: No edge of m can be completely inside C. Suppose there was an edge $f = [a_j, a_{j+1}]$ inside C. By deleting e and f from m and adding either $[a_i, a_j], [a_{i+1}, a_{j+1}]$ or $[a_i, a_{j+1}], [a_{i+1}, a_j]$, we construct a new cycle m' whose distance sequence is strictly smaller than that of m since $d(a_i, a_{i+1}) > \max\{d(a_i, a_j), d(a_{i+1}, a_{j+1}), d(a_i, a_{j+1}), d(a_{i+1}, a_j)\}$. But this is a contradiction since m has the minimum distance sequence.

Therefore, we may assume that no edge of m lies completely inside C. Since e is not in 15-GG there must be at least $w \geq 16$ points of P in C. Let $U = u_1, u_2, \ldots, u_w$ represent these points indexed in the order we would encounter them on the cycle starting from a_i. Let $S = s_1, s_2, \ldots, s_w$ and $T = t_1, t_2, \ldots, t_w$ represent the vertices where s_i is the vertex preceding u_i on the cycle and t_i is the vertex succeeding u_i on the cycle.

Refer to Figure 2 for the following two claims. Let D be the circle centered at a_{i+1}

with radius $2r$.

Claim 2: No point of T can be inside D. Suppose $t_j \in T$ is in D, then $d(t_j, a_{i+1}) < 2r$. Construct a new cycle m' by removing the edges $[u_j, t_j], [a_i, a_{i+1}]$ and adding the edges $[a_{i+1}, t_j], [a_i, u_j]$. Since the two edges added have length strictly less than $2r$, $ds(m') < ds(m)$ which is a contradiction.

Let c be the center of circle C. Let C' be the circle centered at c with radius $2r$.

Claim 3: There are at most 4 points of T in C'. Suppose that there are 5 points of T in C'. Note that the 5 points are in $C' \cap \overline{D}$ by the previous claim. However, this means that there must be two points t_j, t_k such that $\angle(t_j, c, t_k) < \pi/5$. But this implies that $|\overline{t_j t_k}| < 2r$.

Since $|T| \geq 15$, there are at least 11 points of T outside C'. Decompose the plane into 10 cones of angle $\pi/5$ centered at c. By the pigeon-hole principle, there must be one cone with at least 2 points, t_j and t_k. By Lemma 1, $d(t_j, t_k)$ is either less than $2r$ or less than $\max\{d(c, t_j) - r, d(c, t_k) - r\}$. Construct a new cycle m' from m by first deleting $[t_j, u_j], [t_k, u_k], [a_i, a_{i+1}]$. This results in three paths. One of the paths must contain both a_i and either t_j or t_k. Without loss of generality, we suppose that a_i and t_j are on the same path. Add the edges $[a_i, u_k], [a_{i+1}, u_j], [t_j, t_k]$. The resulting cycle m' has a strictly smaller distance sequence since $\max\{d(t_j, u_j), d(t_k, u_k), d(a_i, a_{i+1})\} > \max\{d(a_i, u_k), d(a_{i+1}, u_j), d(t_j, t_k)\}$.

□

3 Size of the Delaunay Graphs

In this section we provide upper and lower bounds on the size of the order-k Delaunay graph. We begin by giving an upper bound on the number of edges of k-DG, i.e., on the number of edges with order at most k in a given set of points. This bound can be derived, as we show next, taking into account the relation of k-DG with higher order Voronoi diagrams [26].

Theorem 2. *Let P be a set of points in general position and let $|k\text{-}DG|$ be the number of edges of the order-k Delaunay graph. Then*

$$|k\text{-}DG| \leq 3(k+1)n - 3(k+1)(k+2).$$

If P is in convex position, then

$$|k\text{-}DG| \leq 2(k+1)n - \frac{3}{2}(k+1)(k+2).$$

Proof. Let b_{pq} be the bisector of points p and q and let $V_k(P)$ be the order-k Voronoi diagram of P. If there exists a circle through p and q containing at most i points, then there exists a segment of b_{pq} which is an edge of $V_{i+1}(P)$. Therefore, an upper bound on the number of bisectors that contribute at least one edge to some Voronoi diagram in the set $\{V_i(P) | 1 \leq i \leq k+1\}$ provides an upper bound on the size of k-DG. To bound this quantity, we start by considering a single bisector b_{pq}. Let B_{pq} be the union of all the edges in the set $\{V_i(P)|1 \le i \le k+1\}$ that lie on b_{pq}. Note that B_{pq} partitions the line b_{pq} into a number of intervals or components. Let λ_k denote the number of components summed over all $\binom{n}{2}$ bisectors. Thus, λ_k is an upper bound on the number of bisectors that contribute at least one edge.

D. T. Lee [23] (see also [9]) showed that the number of edges in the order-k Voronoi diagram is exactly $(6k-3)n - 3k^2 - e_k(P) - 3\sum_{i=1}^{k-1} e_i(P)$ for all $1 \le k \le n-1$, where $e_i(P)$ is the number of subsets of i points of P that can be defined as the intersection of P with a half-plane. Using this, Edelsbrunner et al. [10] prove the following (note that in our definition λ_k refers to $V_{k+1}(P)$):

$$\lambda_k = 3(k+1)n - \frac{3}{2}(k+1)(k+2) - \sum_{i=1}^{k+1} e_i(P).$$

Therefore, the result follows from known bounds on $e_i(P)$: if P is in convex position, then $\sum_{i=1}^{k+1} e_i(P) = (k+1)n$, while $\sum_{i=1}^{k+1} e_i(P) \ge 3\binom{k+2}{2}$ for every set P [10, 24]. $\square$

It is worth remarking that the conjecture by Urrutia [32] stating that every set of n points has a pair such that every circle through them contains at least $\frac{n}{4} - 1$ points in its interior is equivalent to say that k-DG cannot be the complete graph if $k < \frac{n}{4} - 1$.

The previous approach is not useful in providing a lower bound on the size of k-DG because a bisector b_{pq} can have as many as k connected components when all Voronoi diagrams up to order $k+1$ are put together. Therefore, each edge in k-DG may be overcounted as many as k times. Dividing the above bounds by k gives trivial lower bounds since we know that k-DG contains a triangulation of the point set for all $k \ge 0$. Observe that k-DG is the complete graph for $k \ge \frac{n}{2}$ since for every pair of points, there is a circle through that pair of points containing less than half the points. We now prove a slightly stronger than trivial lower bound.

Theorem 3. *Let P be a set of points in general position and let $|k\text{-}DG|$ be the number of edges of the order-k Delaunay graph. Then*

$$|k\text{-}DG| \ge (k+1)n \qquad \text{if } k < \tfrac{n}{2} - 1.$$

Proof. Let $\delta_k(p)$ be the degree of p in the order-k Delaunay graph. We claim that $\delta_k(p)$ is at least $2(k+1)$ for every $p \in P$ and every $k < \frac{n}{2} - 1$. In order to prove the claim, let ℓ by a line through p leaving in both sides at least k points of P. Now consider the two families of circles tangent to ℓ at p ordered according to increasing diameter. If q is the k-th point reached by one of the families, then $pq \in (k-1)$-DG and the claim follows. Now we finish observing that $|k\text{-DG}| = \frac{1}{2}\sum_{p \in P} \delta_k(p)$. $\square$

While general order-k Delaunay graphs are an interesting family of proximity graphs, it should be clear that the low order ones are the most interesting from the viewpoint of applications as they are conceptually the closest to the Delaunay triangulation (see also [21]). This leads us to an improved lower bound for the number of edges of order 1. Before proving this lower bound, we make an observation that follows easily from the definition of 1-*DG*.

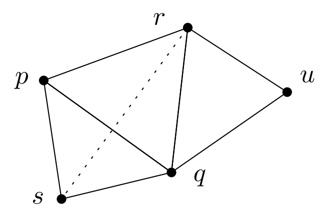

Figure 3: Illustration for the proof of Lemma 2.

Observation 1. Let $DT = 0\text{-}DG$ be the Delaunay triangulation of P.

a) The edge pq has order 1 if and only if $pq \notin DT$ and there exists a point $r \in P$ such that $pq \in DT(P \setminus \{r\})$. In this situation, we say that pq *is generated* by r.

b) If pq is generated by r, then $pr, qr \in DT$. Therefore, an edge pq with order 1 is generated by at most 2 points of P. Furthermore, if pq is generated by 2 points, then it is the diagonal (which is not a Delaunay edge) of a pair of adjacent triangles of DT in convex position.

Lemma 2. *Let $\mathcal{G}$ be the dual graph of DT and let $t_1 = pqr$ and $t_2 = pqs$ be a pair of adjacent triangles of DT in convex position. We say that $t_1 \sim t_2$ if rs has order 1 and is generated by 2 points (which are necessarily p and q). The relation "$\sim$" is a (in general non perfect) matching in $\mathcal{G}$.*

Proof. We have to show that if $t_1 \sim t_2$ then $t_1 \not\sim t_3$ for a triangle t_3 adjacent to t_1. Without loss of generality, we can assume that t_3 has vertices qru as in Figure 3. By contradiction, let us assume that $t_1 \sim t_3$. Then, the edge pu is generated by q and r and, therefore, rs and pu belong to $DT(P \setminus \{q\})$, which is impossible because two edges of $DT(P \setminus \{q\})$ cannot cross. □

Theorem 4. *Let $\Delta_1(P)$ be the number of edges of order 1 of an n-point set P. Then,*

(a) $\Delta_1(P) \geq n - 5$ for every set P.

(b) If P is in convex position, then $\Delta_1(P) \geq \lceil \frac{3n}{2} \rceil - 5$.

Proof. Let us denote by $\delta(p)$ the degree of p in DT. According to Observation 1, the edges of order 1 generated by a point $p \in P$ are exactly the edges in $DT(P \setminus \{p\})$ which are not in $DT(P)$, that is to say, the edges needed to triangulate the hole that appear when p (and the adjacent edges) are removed from $DT(P)$. Therefore, if points are in convex position, every point $p \in P$ generates exactly $\delta(p) - 2$ edges of order 1. On the other hand, from Lemma 2 it follows that the number of edges generated by 2 points is at most $\lfloor n/2 \rfloor - 1$. Then,

$$\Delta_1(P) \geq \sum_{p \in P} (\delta(p) - 2) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) = \left\lceil \frac{3n}{2} \right\rceil - 5$$

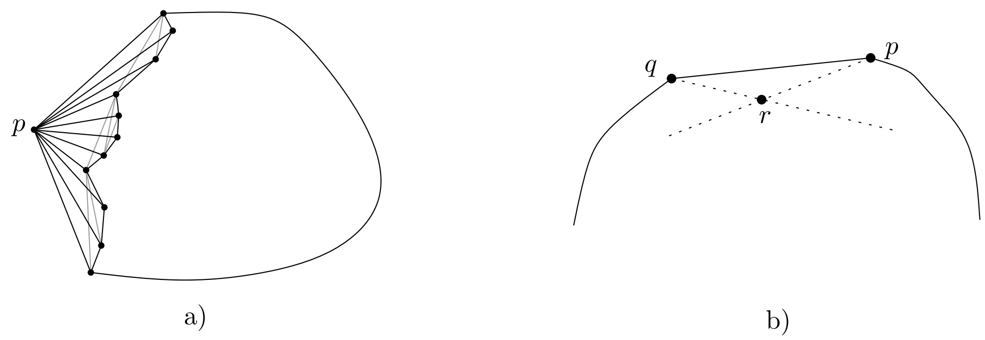

Figure 4: a) Edges generated by p are in grey. b) Point r contributes to $\delta^*(p)$ and to $\delta^*(q)$.

As for (a), we start with the observation that the number of edges generated by any interior point p is $\delta(p) - 3$, but the situation is more complicated if p is a vertex of the convex hull (see Figure 4.a). If we denote by $\delta^*(p)$ the number of neighbors of p in DT which are vertices of the convex hull of $S \setminus \{p\}$, then the number of edges of order 1 generated by p is $\delta(p) - \delta^*(p)$. Let $\mathcal{B}$ and $\mathcal{I}$ be the points of P that are, respectively, on the boundary and in the interior of the convex hull of P. Counting twice the edges of order 1 generated by 2 points we have

$$\sum_{p\in\mathcal{I}}(\delta(p) - 3) + \sum_{p\in\mathcal{B}}(\delta(p) - \delta^*(p)) = 4n - 6 - \big(|\mathcal{I}| + \sum_{p\in\mathcal{B}}\delta^*(p)\big). \tag{1}$$

Finally, consider $t^* = \max\{0, \sum_{p\in\mathcal{B}}\delta^*(p) - 2|\mathcal{B}| - |\mathcal{I}|\}$. We observe that there are at least t^* interior points that contribute twice to $\sum_{p\in\mathcal{B}}\delta^*(p)$. Let r be a point contributing both to $\delta^*(p)$ and to $\delta^*(q)$. Clearly, p and q have to be consecutive vertices of the convex hull and, moreover, if we label the points in such a way that r is on the left of pq as in Figure 4.b, then the unique point to the right of pr is q and the unique point to the left of qr is p. Then the triangle pqr cannot be matched as defined in Lemma 2. Observing that $t^* \leq |\mathcal{B}|$, it follows that

$$\Delta_1(P) \geq 2n - 6 - t^* - \frac{1}{2}(2n - |\mathcal{B}| - 2 - t^*) = n - 5 + \frac{|\mathcal{B}| - t^*}{2} \geq n - 5$$

□

4 Triangulations with order-k edges

In this section we study the combinatorial structure of the set of triangulations that can be constructed with edges of order at most k, i.e., triangulations which are subgraphs of the order-k Delaunay graph. In particular, we study properties of the flip-graph. We begin with a few relevant definitions required for this section.

Definition 1. Let T be a triangulation of P and let prq and pqs be two adjacent triangles in T.

1. The edge pq is *locally Delaunay* if the circle through p, q, and r does not contain the point s.
2. If the quadrilateral $prqs$ is convex, the operation of removing the edge pq and adding the edge rs to T is called a *flip*.

Definition 2. Let us denote by $T_k(P)$ the set of triangulations which are subgraphs of k-DG(P).

1. A flip is *k-legal* provided that the triangulation T prior to the flip and the triangulation T' resulting from the flip are both in the set $T_k(P)$.
2. The order-k flip-graph $FG_k(P)$ is defined as follows: the set of vertices of $FG_k(P)$ are the elements of $T_k(P)$. Two vertices T, T' in $FG_k(P)$ are adjacent if the two triangulations $T, T' \in T_k(P)$ differ by exactly one k-legal flip.

Remark: At this point it may be worth noting that our definition for $T_k(P)$ is *different* from the concept of order k triangulation, as defined by Gudmundsson *et al.* in [14]: a triangulation T is said to have order k if the circumscribing circle of every triangle in T contains at most k points. Clearly, if a triangulation T has order k, then $T \in T_k(P)$, but the converse is not true. Therefore, the number of triangulations in $T_k(P)$ is bigger than the number of order k triangulations, which may be useful if we want to use higher order triangulations in order to optimize some criteria as proposed in [14].

Notice that the flip of an order k-edge is not necessarily k-legal (as happens for example in the Delaunay triangulation). Therefore, the question arises of whether $FG_k(P)$ is connected by flips. We show that if $k = 1$ or P is in convex position then $FG_k(P)$ is always connected and that it may be disconnected otherwise.

Theorem 5. *$FG_1(P)$ is connected for every set P.*

Proof. It is well known that the only triangulation in which every edge is locally Delaunay is the Delaunay triangulation (see [26]), and that starting with any triangulation we always reach the Delaunay triangulation if we keep flipping current non locally Delaunay edges, while that is possible. This can be seen, for example, by lifting the current triangulation to the unit paraboloid and taking the associated polyhedral terrain on the lifted point set. A non locally Delaunay edge of the triangulation gives a reflex edge of the terrain (as seen from below), and its flip corresponds to sticking a tetrahedron below the lifted edge. The terrain keeps "going down" until the lower convex hull of the lifted point set is reached, which corresponds to the Delaunay triangulation of the original point set.

Therefore, to show that $FG_1(P)$ is connected, we can consider an arbitrary triangulation $T \in T_1(P)$ and show that, if T is not the Delaunay triangulation, then there exists a 1-legal flip between an edge $e \in T$ that is not locally Delaunay to an edge e' that is locally Delaunay. By repetition we obtain a path from T to the Delaunay triangulation lying inside $FG_1(P)$.

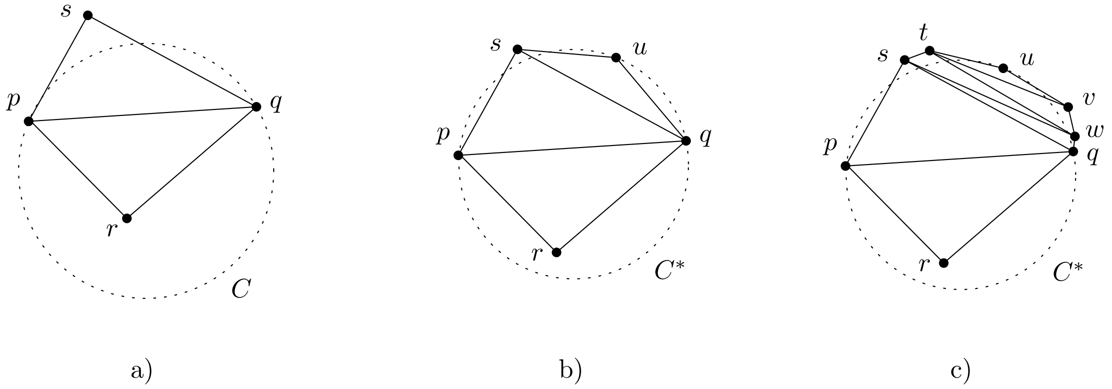

Figure 5: Illustration for the proof of Theorem 5.

Let pq be an edge of T that is not locally Delaunay (then, it has order one) and let rs be the edge that results when pq is flipped. Let C be a witness circle for pq (i.e., a circle with pq on the boundary containing exactly one point). The point contained in C is either r or s. In the following, we assume without loss of generality that r is contained in C (see Figure 5.a).

Let c be the center of circle C and let D be the circle defined by points p, q, s with center d. Consider the family of circles passing through p and q whose centers lie on the line segment cd. In this family of circles, as the center moves from c to d, let C^* be the first circle passing through p, q, and a third point u. The circle C^* contains r in its interior since both C and D contain r in its interior. There are three different cases to consider:

1. If $s = u$, then rs has order zero and we can flip pq to rs.

2. If $qsu \in T$ (Figure 5.b), then pu has order at most one and is locally Delaunay. Therefore, we can flip qs to pu.

3. If $qsu \notin T$ (Figure 5.c) we consider the triangle $tvu \in T$ intersected by pu and its adjacent triangle tvw (also intersected by pu). Without loss of generality, we can assume that v and w lie on the same side of the line defined by the segment pu. Observe that it may happen that edge wt coincides with qs. Any witness circle for tv and for tw contains u, otherwise it would contain the points p and r, which is impossible since the edges have order one. Therefore, the union of tvu and tvw is a convex quadrilateral. Finally, because a witness circle of tw contains u and does not contain v, it follows that uw is locally Delaunay (actually, it is a Delaunay edge) and we can flip tv to uw.

In each of the three cases, we flipped an edge that was not locally Delaunay into an edge that is locally Delaunay with order at most 1. The result follows. $\square$

Theorem 6. *$FG_k(P)$ is not necessarily connected if $2 \leq k \leq (n-3)/3$.*

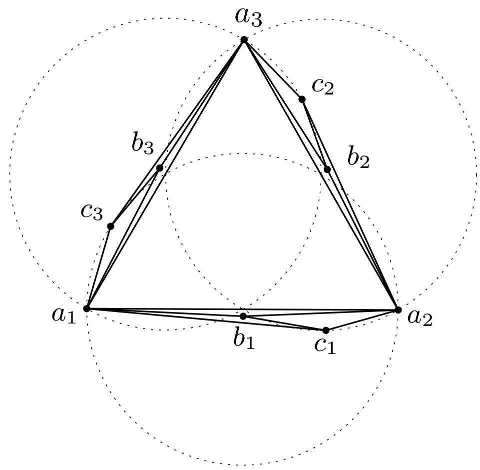

Figure 6: $FG_2(P)$ is not connected.

Proof. We first construct a triangulation $T \in T_2(P)$ on a set P with 9 points and show that no edge of the triangulation permits a 2-legal flip. Therefore, since $FG_2(P)$ has at least two vertices (T and the Delaunay triangulation of P) and T is an isolated vertex of $FG_2(P)$, we see that $FG_2(P)$ is not necessarily connected. We generalize this construction for all $2 \le k \le (n-3)/3$.

We begin the construction of T of a set P with 9 points. Consider an equilateral triangle t with vertices a_1, a_2, a_3 and place points b_1, b_2, b_3 close enough to the midpoints of the edges but outside the triangle t (see Figure 6). Next, consider the circle passing through a_1a_2 and the midpoints of the other two edges of t and place point c_3 on the midpoint of the circular arc joining a_1 with the midpoint of a_1a_3, as shown in Figure 6. Points c_1 and c_2 are placed similarly. It is easy to see that $T \in T_2(P)$ and that T has only three edges that can be flipped, a_1a_2, a_2a_3 and a_3a_1. However, the edges a_2b_3, a_1b_2 and a_3b_1 have order three. Therefore, T is an isolated vertex in $FG_2(P)$.

If we replace each point b_i with a set of $k-1$ points arbitrarily close and consider the Delaunay triangulation of P constrained by the edges a_1a_2, a_2a_3 and a_3a_1 then we get an isolated vertex in FG_k.

□

We conclude this section by showing that FG_k is always connected if P is in convex position. We begin with a few helpful lemmas. Recall that $\mathrm{o}(pq)$ denotes the order of the edge pq.

Lemma 3. *Let pq be an edge with order $k > 0$ and let C be a witness circle for pq.*

(a) If $u \in C$ then $\mathrm{o}(up) < k$ and $\mathrm{o}(uq) < k$.

(b) If $u, v \in C$ then $\mathrm{o}(uv) \le k-2$.

Proof. It both cases, it is clear that we can shrink C and get a circle containing less than k points for part (a) and at most $k-2$ points for part (b). □

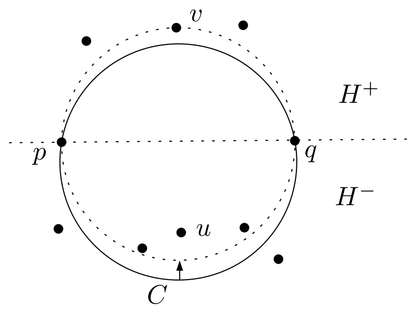

Figure 7: Illustration for the proof of Lemma 4.

Let pq be an edge of T with order $k > 0$ and let H be one of the halfplanes defined by the line supporting pq. We say that u is the *first point* of pq in the half plane H if $u \in H$ and the circle through p, q and u does not contain any point of $P \cap H$.

Lemma 4. *Let pq be an edge with order $k > 0$ and let u and v be the first points of pq (one on each half plane). Then we have:*

(a) $\mathrm{o}(uv) < k$.

(b) If $\mathrm{o}(vp) = k$*, then* $\mathrm{o}(up) < k$ *and* $\mathrm{o}(uq) < k$.

Proof. Let C be a witness circle for pq. Because $\mathrm{o}(pq) \geq 1$, C contains at least one of the first points u or v. Without loss of generality assume it is u. We distinguish two cases:

– If $u, v \in C$ from Lemma 3.b it follows that $\mathrm{o}(uv) \leq k - 2$.

– $u \in C$ and $v \notin C$ (see Figure 7). In this case, we observe that if we consider circles passing through p and q and "moving" towards v, we reach v before any other point enters or leaves the circle and, therefore, we get a circle passing through p, q and v, containing u and $k - 1$ other points of the half plane. Therefore, $\mathrm{o}(uv) \leq k - 1$.

For the second part of the lemma, we observe that if $\mathrm{o}(vp) = k$, then from Lemma 3 it follows that $v \notin C$. Therefore, we can repeat the procedure of the previous paragraph and get a circle passing through p, q and v and containing u and $k - 1$ other points in its interior, showing that $\mathrm{o}(up) < k$ and $\mathrm{o}(uq) < k$. □

We are now ready to proof the following theorem.

Theorem 7. *If P is in convex position, then $FG_k(P)$ is connected for every $k \geq 0$.*

Proof. Let $T \in T_k(P)$ be a triangulation with exactly m order-k edges. We want to find a sequence of k-legal flips that converts T into a triangulation $T' \in T_k(P)$ having strictly less than m order-k edges. By iterating this process we end up with the Delaunay triangulation, thereby showing that FG_k is connected.

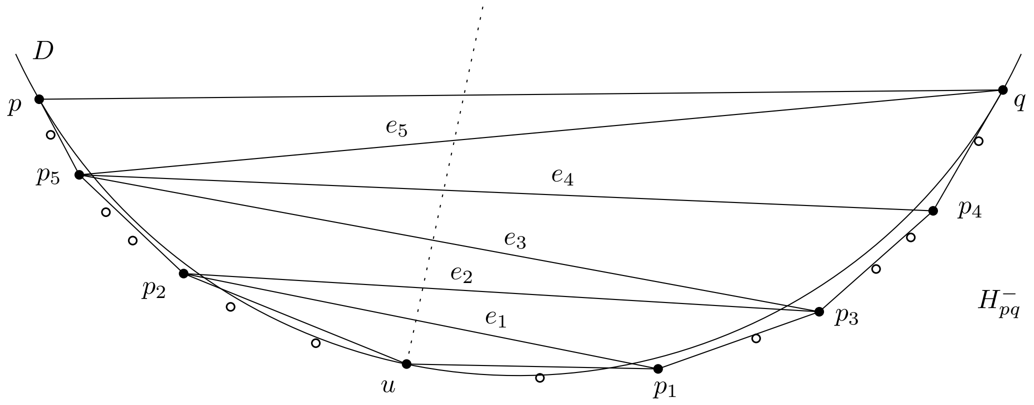

Figure 8: Illustration for the proof of Theorem 7.

Let pq be an edge of T with order k. We assume that the edge pq is horizontal and denote by H_{pq}^+ and H_{pq}^- the upper and the lower half planes defined by the line containing pq. Finally, let u and v be the first points of pq in H_{pq}^- and H_{pq}^+, respectively. We start by finding a sequence of legal flips in FG_k such that pqu and pqv are adjacent triangles in the triangulation. Let D be the circle passing through p, q and u. Because u is the first point in H_{pq}^-, we know that D does not contain any point of $P \cap H_{pq}^-$. Let $e_1, \ldots, e_j$ be the set of edges of T intersected by uv and contained in H_{pq}^-, ordered from u to pq. Either p or q is an endpoint of e_j and, without loss of generality, we can assume that $e_j = p_j q$ and that $up_1p_2 \in T$, if we label the vertices suitably (see Figure 8).

Let C_i be a witness circle for e_i. We claim that $u \in C_i$ for $i = 1, \ldots, j$. In order to prove the claim, observe that if $u \notin C_i$, then $p, q \in C_i$. But then we get a contradiction because from Lemma 3 it follows that $\mathrm{o}(pq) < \mathrm{o}(e_i) \leq k$.

Because $u \in C_i$, Lemma 3.a guarantees that $\mathrm{o}(up_i) < k$ for $i = 1, \ldots, j$ and $\mathrm{o}(uq) < k$. However, we can only guarantee that $\mathrm{o}(up) \leq k$. In fact, if there is no witness circle of pq containing u, then D contains k points (all of them in H_{pq}^+) and it may happen that $\mathrm{o}(up) = k$. Now we remove the edges $e_1, \ldots, e_j$ and insert the edges $up_3, \ldots, up_j, uq, up$ by a sequence of consecutive flips that correspond to a path inside FG_k. Thus, all new flipped edges, except perhaps up, have order strictly smaller than k.

We repeat the procedure in the upper half plane and introduce edges $vq_3, \ldots, vq_h, vp, vq$ (again, all of them have order smaller than k, except perhaps vp or vq). Finally, we flip the edge pq to uv, which has order smaller than k according to Lemma 4.b.

Let us count the number of order-k edges in the resulting triangulation. We claim that at most one new edge with order k has been inserted. As observed before, only one new edge in each half-plane can have order k. However, observe that if up, which is the unique edge in H_{pq}^- which could have order k actually reaches this bound, then from Lemma 4.b it follows that $\mathrm{o}(vp) < k$ and $\mathrm{o}(vq) < k$, thereby proving the claim.

So far, we have removed at least one edge with order k, namely pq, and have added at most one new edge with order k. If the number of edges with order k has decreased, the proof is finished, so suppose that the number of order k edges remains the same. Without loss of generality, we can assume that $\mathrm{o}(up) = k$. In this situation, a witness circle for pq does not contain u (the opposite would contradict Lemma 3.a). Therefore, the circle D through p, q, and u contains k points (all of them in H^+_{pq}). But then, if we repeat the whole process, we know that all the edges inserted in H^+_{up} have order smaller than k. This shows that in this iteration either the number of order k edges decreases or the order k edge which is inserted is contained in H^-_{up}. Because $P \cap H^-_{up} \subsetneq P \cap H^-_{pq}$, we cannot run into the case of maintaining the number of order k edges indefinitely.

□

5 An application of higher order Delaunay graphs

Given a set of n points in the plane, Har-Peled and Smorodinsky [17] showed how to assign one of m colors to each of the n points such that every circle C containing more than one point has the property that at least one of the points in C has a unique color. Such a coloring is called a *conflict-free* coloring (CF-coloring for short). The Delaunay graph of the n points is used in the coloring algorithm and also to show that m is $O(\log n)$. Har-Peled and Smorodinsky show that this type of coloring finds application in the assignment of frequencies in a cellular network. Let each color represent a distinct frequency. Let the n points in the plane represent communication towers. Each tower is assigned a communication frequency. When a client (i.e., cell phone) needs to communicate with a tower, it searches for all towers within its communication range which is represented by the circle C. Two towers communicating at the same frequency in the phone's range create interference. Therefore, to ensure good reception, it is desirable to always have at least one tower within range communicating at a unique frequency. Their result implies that $O(\log n)$ frequencies suffice to ensure this property. In this section, we generalize the result in [17]. Recall that the maximum number of edges in $(k-1)$-DG is ckn for some constant c. We show that with $\log n/(\log((2ck)^2 - 1) - \log((2ck)^2 - 2))$ colors, a set of n points in the plane can be colored so that every circle containing at least k points contains at least k points with unique color. We call such a coloring a k-conflict-free coloring. In the context of cellular networks, this can be viewed as ensuring that for every client in range of k or more towers, there always exists at least k different towers with which the client can communicate without interference.

As noted in Theorem 2, the number of edges in $(k-1)$-DG is at most ckn where $c = 3$ when the points are in general position and $c = 2$ when points are in convex position. This implies that the average degree of a vertex in $(k-1)$-DG is at most $2ck$.

Lemma 5. *Every $(k-1)$-DG has an independent set of size at least $n/((2ck)^2 - 1)$ where each vertex in the set has degree at most $2ck$.*

Proof. The minimum degree in $(k-1)$-DG is 2 and the average degree is at most $2ck$. Let V and E be the vertex and edge set of $(k-1)$-DG. Let x be the number of vertices in $(k-1)$-DG with degree at most $2ck$. Since $2|E| = \sum_{v \in V} \deg(v)$, we have that $2ckn \geq 2x + (n-x)(2ck+1)$. This implies that $x \geq n/(2ck-1)$. Since there are at least $n/(2ck-1)$ vertices with degree at most $2ck$, there is an independent set of size at least $n/((2ck-1)(2ck+1))$ where each vertex has degree at most $2ck$. □

The coloring algorithm is virtually identical to the algorithm given in [17]

Algorithm 1 k-conflict free coloring of set S of n planar points

Set $i = 0$, where i denotes an unused color.
while $|S| \neq 0$ **do**
  Let I be an independent set of size at least $n/((2ck)^2 - 1)$ in $(k-1)$-DG(S).
  Color all points in I with color i.
  Remove I from S.
  Increment $i = i + 1$.
end while

In the next lemma, we show that the above algorithm provides a k-conflict free coloring and the total number of colors used is $\log n/(\log((2ck)^2 - 1) - \log((2ck)^2 - 2))$

Lemma 6. *With* $\log n/(\log((2ck)^2 - 1) - \log((2ck)^2 - 2))$ *colors, a set of* n *points can be colored so that every circle containing at least* k *points contains* k *points whose color is unique.*

Proof. First, at each iteration, we remove an independent set of size at least $n/((2ck)^2 - 1)$. Let $d = (2ck)^2 - 1$. Let $C(n)$ represent the number of colors used for a $(k-1)$-DG graph with n vertices. We can bound $C(n)$ with the following recurrence: $C(n) \leq C((d-1)n/d) + 1$. This recurrence resolves to $C(n) \leq \log n/(\log d - \log(d-1))$ as required.

Next, we show that the coloring is k-conflict free. Let C be any circle containing a set P of at least k points. Consider the k points in C whose colors have highest value (recall that the first independent set was given color 0 and an independent set removed at step i was given color i). If all these k points have unique colors, the lemma is proved. For sake of a contradiction, assume that at least 2 of these k points have the same color. Let i be the largest color whose value is not unique. Note that there are fewer than k points in P whose color value is strictly greater than i. Also note that at iteration i of the algorithm, all points with color less than i have been removed from P. Let P_i be the set of points in P receiving color i. Since C contains P_i, there is a circle C' contained in C that has two points x, y of P_i on its boundary and no points of P_i in its interior. However, since there are fewer than k points whose color is larger than i, this means that C' contains fewer than k points in its interior at iteration i of the algorithm. However, this contradicts the fact that x and y are in an independent set selected at iteration i. □

Corollary 1. *A set of* n *points in general position can be colored with* $\log n/(\log((6k)^2 - 1) - \log((6k)^2 - 2))$ *colors so that every circle containing at least* k *points contains* k *points whose color is unique. If the points are in convex position, then* $\log n/(\log((4k)^2 - 1) - \log((4k)^2 - 2))$ *colors are sufficient*

Note that we only used the fact that there is a large number of vertices of bounded degree in $(k-1)$-DG in order to show that there is a sufficiently large independent set. If one can find a larger independent set that is guaranteed to exist in all $(k-1)$-DG graphs, then the preceding bounds can be improved.

6 Concluding remarks and open problems

In this paper, we have established several properties of the higher order Delaunay graphs, yet many questions remain open. Among these, finding the minimum value for which k-DG is always Hamiltonian is an especially intriguing problem. Although we have only been able to prove Hamiltonicity for 15-GG (and hence 15-DG), we believe that 1-DG is Hamiltonian.

On the other hand, finding tight bounds for the size of the graphs is a hard problem in which progress may come from different directions related to this issue.

We believe, as also mentioned in [21], that triangulations that use lower order triangles or edges –especially 1st order– are the most interesting, as they are the slightest departure from the Delaunay triangulation. However, finding the best 1st order triangulation is an NP-hard problem for many criteria, as proved in [21]. Hence another natural future direction of research made possible by our connectivity result is to experiment with the flip heuristics for triangulations that are subgraphs of 1-DG. This direction has already been considered in [21] for the case in which the focus is on triangles more than on edges.

Finally, let us comment on our assumption throughout the paper that our point sets contain no collinearities and no cocircularities. When that is not the case, one may assume that in a first step degeneracies have been removed by using some perturbation scheme (see [11], [33] and [12]). A second option would be to look for a canonical way of defining the graphs that leads to no ambiguity, as suggested by Edelsbrunner [9] for the Delaunay triangulation via the *globally equiangular triangulation*, which can also be computed in $O(n \log n)$ time [25]. For the 0-order Delaunay case an option that has also been considered in the literature is to say that two points $p, q \in P$ are adjacent when there is some circle through p and q whose interior is empty of points from P. However, notice that in this case a set of cocircular points such that the circle through them contains no other point of P defines a complete subgraph of this Delaunay graph, that hence can have quadratic size. For the problems in this paper, the second and third options require a complete reformulation of many of our results and proofs, or simply make no sense, as is the case for the upper bounds on the size of the graphs. Hence, we prefer to defer these explorations to future research and avoid obscuring the comprehension of the basic concepts.

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