

On Local Transformations in Plane Geometric Graphs Embedded on Small Grids*

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Abstract

Given two n -vertex plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|E_1| = |E_2|$ embedded in the $n \times n$ grid, with straight-line segments as edges, we show that with a sequence of $O(n)$ point moves (all point moves stay within a $5n \times 5n$ grid) and $O(n^2)$ edge moves, we can transform the embedding of G_1 into the embedding of G_2 . In the case of n -vertex trees, we can perform the transformation with $O(n)$ point and edge moves with all moves staying in the $n \times n$ grid. We prove that this is optimal in the worst case as there exist pairs of trees that require $\Omega(n)$ point and edge moves. We also study the equivalent problems in the labeled setting.

1 Introduction

Informally, a local transformation is an operation performed on the vertices and edges of a graph. The term local is used because generally the operation does not affect the whole graph. Typically, the vertices of the graph affected by a local transformation are the neighborhood of a constant number of vertices. For example, an edge contraction* is a local transformation that affects the neighborhood of two vertices.

Once a local transformation has been defined, properties and applications of the local transformation with respect to a given class of graphs are studied [3–6, 9–12, 14, 16–20, 23]. A natural question with respect to local transformations is:

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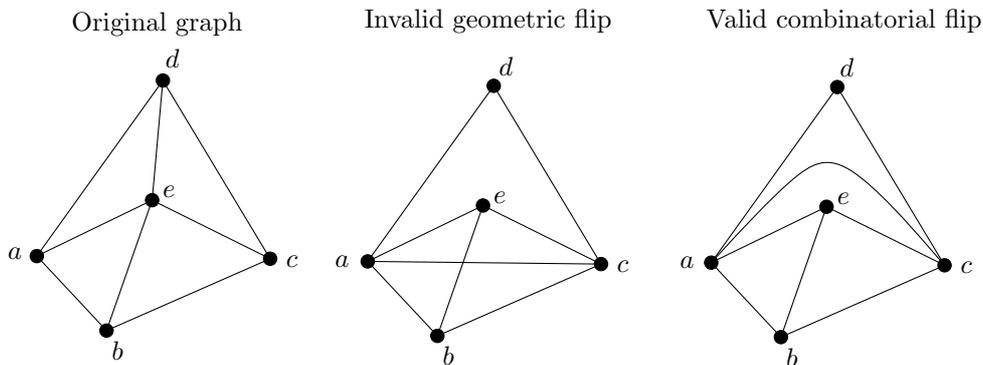
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*As there are many different flavors of edge contractions, for this example, we assume that an edge contraction in a simple graph is the operation of deleting an edge and merging the two endpoints of the edge into a new vertex whose neighborhood is the union of the neighborhoods of the two endpoints. All multiple edges are removed so that the resulting graph is still simple.

Does performing a local transformation of a graph in a given class keep the graph in the same class? For example, if the class is bipartite graphs, then performing an edge contraction on a bipartite graph does not necessarily keep it bipartite. However, if the class of graphs is complete graphs, then edge contractions do keep the graph in the same class. Another question that is often studied is: Given an instance of a graph of a particular class, can all the graphs of this class be enumerated via local transformations on the given instance?

The local transformation that initiated this study is referred to as an *edge flip* or more generally an *edge move*. The class of graphs on which edge flips are defined is triangulations (and sometimes *near-triangulations*[†]) and edge moves are normally defined on planar graphs. The operation of an edge flip or edge move is simply the deletion of an edge, followed by the insertion of another edge such that the resulting graph remains planar and simple. More formal definitions are given in the next section.

Wagner [22] proved that given any two triangulations $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2| = n$, there always exists a finite sequence of edge moves that transforms G_1 into a graph $G_3 = (V_3, E_3)$ that is isomorphic to G_2 . That is, there exists a mapping $\phi : V_2 \rightarrow V_3$ such that for $u, v \in V_2$, $uv \in E_2$ if and only if $\phi(u)\phi(v) \in E_3$. Subsequently, Komuro [13] showed that in fact $O(n)$ edge flips suffice. Recently, Bose *et al.* [2] showed that $O(\log n)$ simultaneous edge flips suffice and are sometimes necessary. This setting of the problem is referred to as the combinatorial setting since the triangulations are only embedded combinatorially, i.e. the cyclic order of edges around each vertex is defined.



Delete edge de and add edge ac . This is a valid combinatorial edge move since the graph is still planar but it is an invalid geometric edge move since the edges ac and be intersect properly.

Figure 1: Valid combinatorial edge flip but invalid geometric edge flip.

In the geometric setting, a triangulation or near-triangulation is embedded in the plane such that the vertices are points and the edges are straight-line segments. Henceforth, we only consider graphs embedded in the plane having straight-line segments for edges. Edge flips and edge moves are still valid operations in this setting, except that now the edge that is added must be a

[†]A near triangulation is a plane graph where every face except possibly the outerface is a triangle

line segment and this line segment cannot properly intersect any of the existing edges of the graph. This additional restriction implies that there are valid edge moves in the combinatorial setting that are no longer valid in the geometric setting since even though the graph resulting after a move is planar and simple, it does not have a plane embedding. See Figure 1.

Lawson [15] showed that given any two near-triangulations N_1 and N_2 embedded on the same n points in the plane, there always exists a finite sequence of edge flips that transforms the edge set of N_1 to the edge set of N_2 . Hurtado *et al.* [12] showed that $O(n^2)$ flips are always sufficient and sometimes necessary. Subsequently, Galtier *et al.* [9] showed that $O(n)$ simultaneous edge flips are sufficient and sometimes necessary.

Note that there is a discrepancy between the combinatorial result and the geometric one. In the combinatorial setting, Wagner [22] showed that every triangulation on n vertices can be attained from every other triangulation via edge flips. In the geometric setting, Lawson [15] showed that only the near-triangulations that are defined on the specified point set can be attained via edge flips. For example, in the point set shown in Figure 2, no planar K_4 (complete graph on 4 vertices) can be drawn on the given point set without introducing a crossing. In fact, in the geometric setting, given a set of points in convex position, the only plane graphs that can be drawn without crossing are outer-planar.

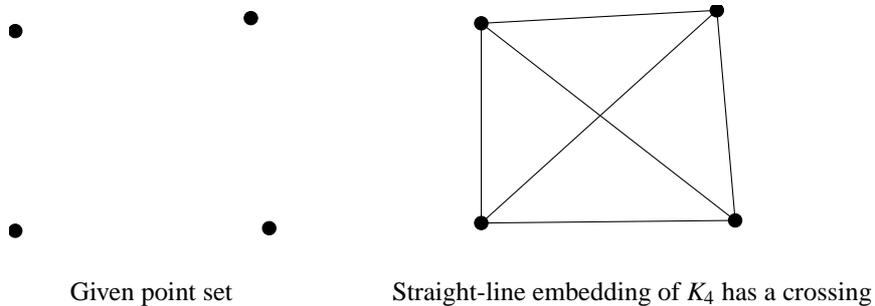


Figure 2: Discrepancy between combinatorial and geometric setting.

It is precisely this discrepancy that sparked our investigation. The first question we asked is whether or not there exists a simple local transformation that permits the enumeration of all n -vertex triangulations in the geometric setting. In order to answer this question, the local transformation must be more general than an edge move. Two key ingredients need to be specified for this question. First, we need to specify the set of points P on which these graphs are embedded and on which the transformations can be performed. To overcome the discrepancy with the combinatorial setting, this set of points must have the property that every n -vertex triangulation has a straight-line embedding on an n -point subset of P . Such a set of points is called a *universal point set*. Schnyder [21] showed that the $n \times n$ grid is a universal point set for all n -vertex planar graphs (see also [8]). Therefore, a grid is a natural choice for this setting. However, using a grid comes at a cost since there are many collinear points in a grid, we need to deal specifically with degeneracies. Despite this obstacle, we use grids as our universal point set and we outline the exact grid sizes required for our results. All of our grid sizes are within a constant of the optimal for straight-line

embeddings of planar graphs. It is important to keep the grid size as small as possible since large grids hinder practical applications of these transformations.

The second ingredient is the set of allowable local transformations. Besides the edge move, the other local transformation we use is a *point move*. A point move is simply the modification of the coordinates of one vertex. The move is valid provided that after moving the vertex to a new grid point, no edge crossings are introduced.

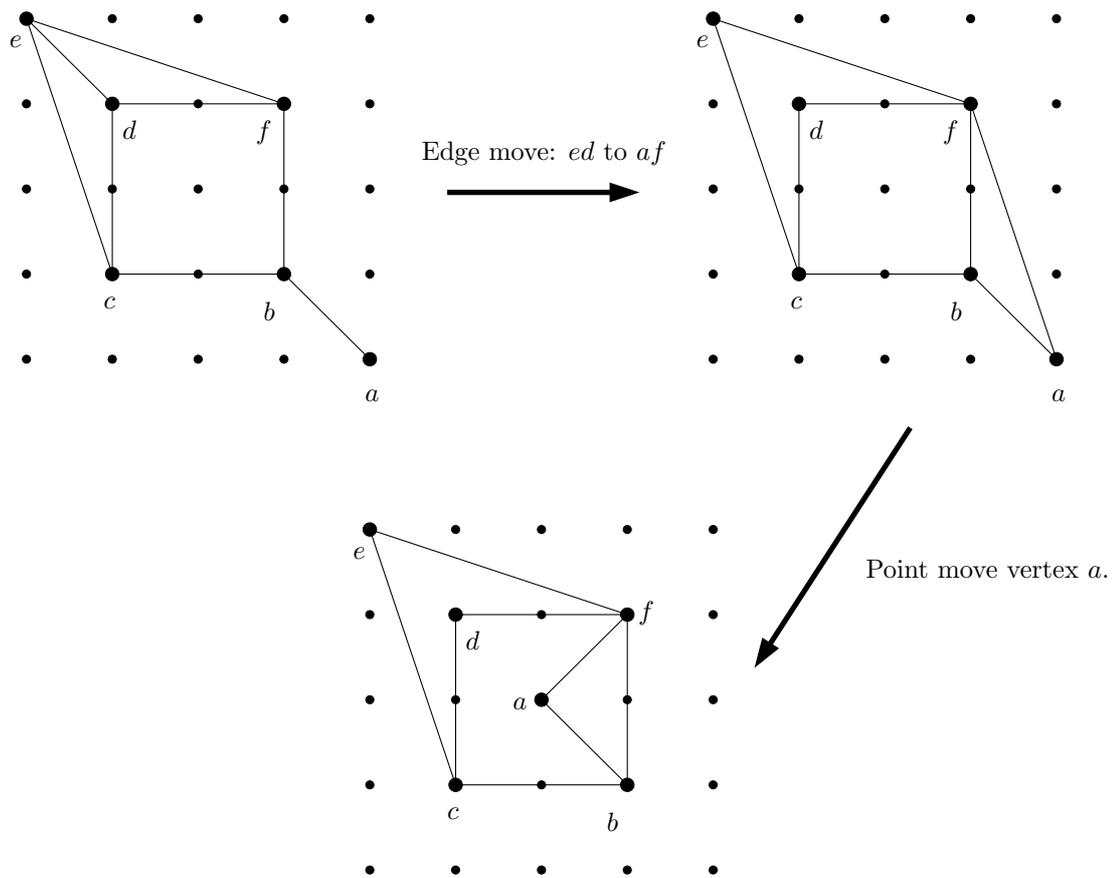


Figure 3: Example of an edge move followed by a point move

2 Transforming One Plane Triangulation to Another

In this section, we show that $O(n)$ point moves and $O(n^2)$ edge moves suffice to transform one plane n -vertex triangulation into another. The first temptation is to simply try to mimic a combinatorial flip in the geometric setting with point and edge moves. However, this approach proved to be quite difficult to characterize the point and edge moves needed to mimic one combinatorial flip. In our proof, we draw ideas from Wagner's original result without mimicking combinatorial flips. Essentially, we show how to transform any given plane triangulation into a canonical one, which immediately implies the result.

More precisely, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2| = n$ be two triangulations embedded in an $n \times n$ grid. Let the origin $(0, 0)$ of this grid be the bottom left corner. Let P be a $5n \times 5n$ grid with bottom left corner located at $(-2n, -2n)$ and top right corner located at $(3n, 3n)$. During the whole sequence of moves, the location of every point move is a grid point of P (i.e. P is our universal point set). We show how to construct a sequence of $O(n^2)$ edge moves and $O(n)$ point moves that transforms both G_1 and G_2 into a canonical form. The canonical triangulation is a triangulation where the outer face consists of vertices located at $(-2n, -1)$, $(3n, -1)$, and $(\lceil n/2 \rceil, 3n)$. The other $n - 3$ vertices are located at $(\lceil n/2 \rceil, 3n - i)$, $1 \leq i \leq n - 3$. The two bottom corner vertices are stars adjacent to all other vertices and the graph induced by the remaining vertices is a path which we will call the *spine*. The canonical triangulation is shown in Figure 4. Note that once we have an embedding of the canonical triangulation, the actual location of the coordinates is no longer important because with $O(n)$ point moves, it is fairly simple to move from one embedding of the canonical triangulation to any other as long as the spine is on a vertical grid line and the outer-face forms a triangle.

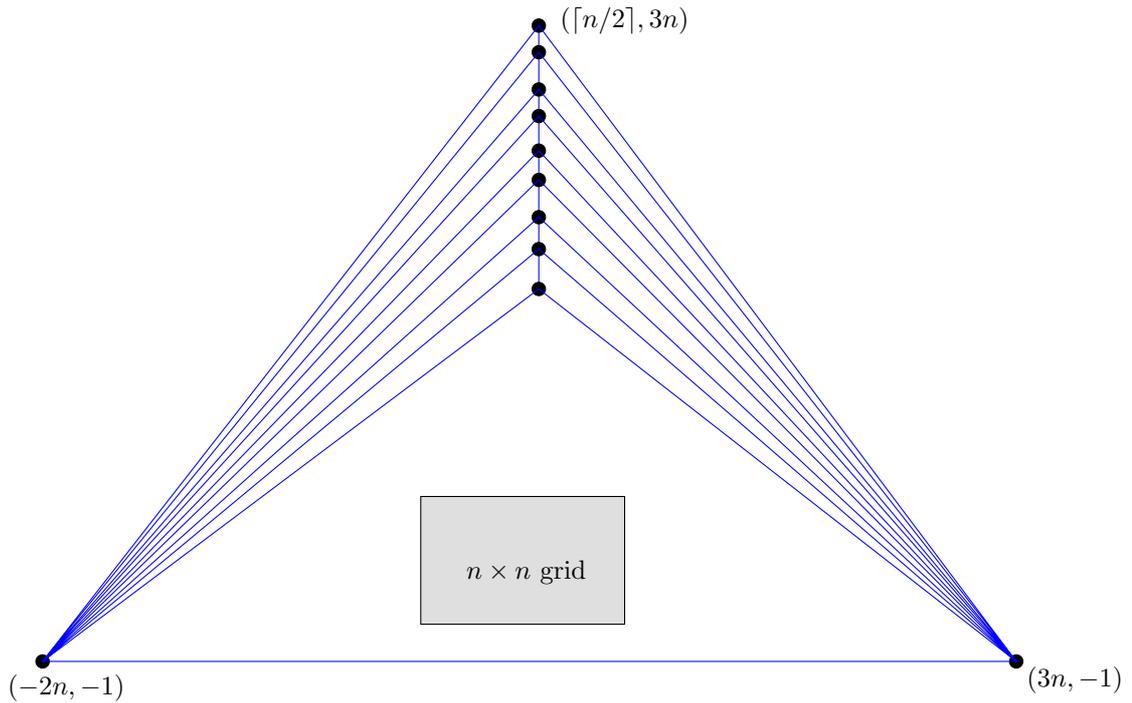


Figure 4: Illustration of the canonical triangulation and the initial grid.

Before showing how to construct the sequence of point and edge moves, we need to establish a few basic building blocks. One useful tool is the result by Hurtado *et al.* [12]

Theorem 1. [12] *Let T_1 and T_2 be two triangulations whose vertex set is a set of n points in the plane. With $O(n^2)$ edge moves, T_1 can be transformed into T_2 .*

However, a key technical lemma in their proof of Theorem 1 is an essential tool in our work.

Lemma 1. [12] Let $T = (V, E)$ be an arbitrary near-triangulation whose vertex set is a set of n points in the plane. Let a, b, c be three consecutive vertices on the outerface of T . Let P be the path from a to c on the convex hull[‡] of $V \setminus b$. With precisely k edge moves, where k is the number of edges of T that intersect P , we can transform T into a triangulation that contains P . Note that k is $O(n)$.

Next, we observe two simple facts about triangles that will be helpful in the sequel.

Observation 1. Let $\triangle(abc)$ be a triangle in the plane. Let x be a point contained in the interior of the triangle. Any line through x which has both b and c in one half-plane must have a in the other and must intersect the line segments ab and ac .

Observation 2. Let $a = (0, 0)$, $b = (x_1, y_1)$ with $x_1, y_1 > 0$, $c = (x_1, y_1 + 1)$ and $d = (x_2, y_2)$ with $x_2 > x_1$ and $y_2 \geq (y_1 + 1)x_2/x_1$. The point c is contained in the interior of triangle $\triangle(abd)$.

We now describe a sequence of edge moves and one point move which we will call an *apex slide*. The setting for an apex slide is the following. Let a, b, c be the vertices of a 3-cycle in triangulation G (i.e. $\triangle(abc)$ is either a face of G or a separating triangle in G). Let x be a point such that xbc forms a triangle with both x and a are on the same side of a half-plane defined by the line through bc . Let D be all the vertices of $G \setminus a$ in $\triangle(abc)$. The set D is contained in $\triangle(xbc)$, and every edge in G intersected by segment xb or xc is adjacent to a . See Figure 5 for an illustration.

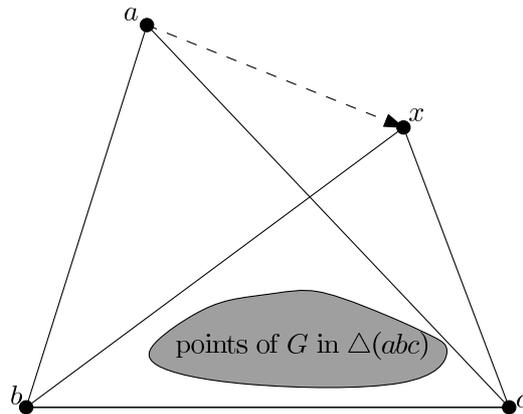


Figure 5: Apex Slide

Lemma 2. With $O(n)$ edge moves and one point move, vertex a can be moved to point x .

Proof. Let D be the vertices of $G \setminus a$ in $\triangle(abc)$. Let $C = c_0, c_1, \dots, c_k$ be the clockwise order of the convex hull of D starting at $b = c_0$ and ending at $c_k = c$. By Lemma 1, with $O(n)$ edge moves, we can convert the triangulation contained in

[‡]The convex hull of a set of points is the smallest convex polygon containing the set. See any standard reference in Computational Geometry [7] for an overview.

abc to one which contains the segments ac_i and the edges of C . Once this is accomplished, Observation 1 implies that we can move a to x without introducing any crossing since C is contained in both $\triangle(abc)$ and $\triangle(xbc)$ and by construction no other edge of G intersects xb or xc . Thus, a total of $O(n)$ edge moves and 1 point move suffice as required. \square

To initiate the whole process, we need to show how given a triangulation embedded in the $n \times n$ grid we can always move the vertices of its outerface to the coordinates $(-2n, -1)$, $(3n, -1)$ and $(\lceil n/2 \rceil, 3n)$. In order to continue, we need to define the *wedge* of a convex vertex. Let C be a convex polygon on n vertices. Let v be a vertex of C , and e_1, e_2 the two edges of C adjacent to v . Given the line l_1 (resp. l_2) defined by the endpoints of e_1 (resp. e_2), let h_1 (resp. h_2) be the half-plane defined by this line that does not contain the interior of C . The *wedge* of v denoted $W(v)$ is $h_1 \cap h_2$. The angle of a wedge is the angle between the two rays defining the boundary of the wedge. The property of a wedge that we will exploit throughout is the following: given a $\triangle(abc)$, let x be a point in $W(a)$. Note that $\triangle(abc) \subset \triangle(xbc)$. This containment property allows one to use apex slides.

Lemma 3. *Given a triangulation $G = (V, E)$ embedded in the $n \times n$ grid, with $O(n)$ edge moves and at most 8 point moves, we can transform it into a triangulation whose outerface has coordinates $(-2n, -1)$, $(3n, -1)$ and $(\lceil n/2 \rceil, 3n)$. All other vertices of G have coordinate values between 0 and n (i.e. they are in the original $n \times n$ grid).*

Proof. Let $V = v_1, v_2, \dots, v_n$ be labeled such that v_1, v_2 and v_3 are the vertices on the outerface. Let the coordinates of each v_i be (x_i, y_i) with $0 \leq x_i, y_i \leq n$. Since the outerface is a triangle, the sum of the angles of the wedges of its vertices is π . This means that one of its vertices permits an apex slide vertically (either up or down) and one of its vertices permits an apex slide horizontally (either left or right). Note that these can be the same vertex. Without loss of generality, assume that v_1 can be moved left. All other cases are symmetric.

The fact that v_1 can be moved left means that the intersection of a horizontal line through v_1 with $W(v_1)$ is a ray rooted at v_1 pointing left. Move v_1 to $(-2n, y_1)$. After this move, notice that the angle of the wedge at v_1 is strictly less than $\pi/2$. This implies that v_1 cannot be moved vertically. Therefore, one of v_2 or v_3 can be moved vertically. Assume, without loss of generality, that v_2 can be moved down. Again, all other cases are symmetric. Move v_2 to $(x_2, -2n)$ each with an apex slide. This amounts to $O(n)$ edge moves and 2 point moves. Next, apply two apex slides to move v_1 to $(-2n, n+1)$ and v_2 to $(n+1, -2n)$. Both these moves can be applied since by construction no segment with one endpoint on the segment $[(-2n, -1), (-2n, n+1)]$ and the other endpoint on segment $[(-1, -2n), (n+1, -2n)]$ intersects the original $n \times n$ grid.

Once these moves have been applied, the outerface is such that v_3 allows an apex slide up to $(x_3, 3n)$. Once this has been accomplished, three more apex slides place the three vertices in canonical position. A total of 8 apex slides were used giving the desired result. \square

We now describe the main step in the process. Let a, b, c be the vertices of the outerface of a triangulation G embedded on a grid, such that b and c lie on

the same horizontal grid line L_1 , there are at least $5n - 1$ grid points between b and c , the vertex a is above b and c . Let x be a point of the grid that is not a vertex of G such that a and x lie on the same vertical grid line L_2 such that the grid point $z = L_1 \cap L_2$ is between b and c with at least $2n$ grid points between b and z and at least $2n$ grid points between c and z . The triangle $\triangle(abc)$ is the outerface of G and the point x be strictly inside triangle $\triangle(abc)$. All other vertices of G are strictly inside triangle $\triangle(bxc)$. There are at least n grid points on the segment ax . Notice that if we are given a triangulation in an $n \times n$ grid and apply Lemma 3, then we meet the conditions specified above.

Lemma 4. *With $O(n^2)$ edge moves and $O(n)$ point moves, we can transform G into canonical form.*

Proof. We proceed by induction on the number h of vertices of G in $\triangle(bxc)$.

Base Case: $h = 0$. The lemma holds trivially since no moves are required.

Inductive Hypothesis: $0 \leq h \leq k, k > 0$. Assume that $d_1 h^2$ edge moves and $d_2 h$ point moves suffice with constants d_1 and d_2 .

Inductive Step: $h = k + 1$. Let r be the first grid point below a . Let $C = c_0, c_1, \dots, c_{m+1}$ be the clockwise order of the convex hull of the vertices of $G \setminus a$ starting at $b = c_0$ and ending at $c = c_{m+1}$. Apply Lemma 1, to convert G to a triangulation containing C and all segments ac_i for vertices c_i of the convex hull. This is accomplished with $d_3 k$ edge moves for constant d_3 .

Let $c_j c_{j+1}$ be the edge of the convex hull that intersects the vertical line through ax . If the line through ax contains a vertex of the convex hull, assume this vertex is c_j . There are two cases to consider.

Case: One of c_j or c_{j+1} is a vertex of the convex hull. Assume without loss of generality that c_j is a convex hull vertex. Since c_j is a vertex of the convex hull, the points c_{j-1}, c_j , and c_{j+1} are not collinear. Since the grid point r is in triangle $\triangle(ac_{j-1}c_{j+1})$ by construction, we can apply an apex move to move point c_j to r .

Case: The edge $c_j c_{j+1}$ is in the interior of the convex hull edge $e = c_s c_t$. Assume for the moment that the edge e has positive slope. Since c_s is a vertex of the convex hull, this means that c_{s-1}, c_s and c_{s+1} are not collinear. By Observation 2, there is a grid point y one unit vertically above c_s inside triangle $\triangle(ac_{s-1}c_s)$. Apply an apex move to move c_s to y . This removes the collinearity from the convex hull. Now the edge yc_t is on the convex hull. Recompute the convex hull and apply Lemma 1 so that a is adjacent to all edges of the convex hull. Now we have reduced the situation back to the previous case. A symmetric argument holds if e has negative slope.

Therefore, with $d_4 k$ edge moves and at most 2 point moves, we remove one vertex of G from $\triangle(bxc)$, and move it to r . Now, there are only k vertices of G remaining in the triangle $\triangle(bxc)$. Apply Lemma 1 so that r is adjacent to all vertices on the convex hull of $G \setminus \{a, r\}$. We can now apply the inductive hypothesis. The total number of edge moves is $d_1 k^2 + d_4 k$ and the total number of point moves is $d_2 k + 2$. If we set $d_1 > d_4$ and $d_2 > 2$, then $d_1 k^2 + d_4 k < d_1 (k + 1)^2$ and $d_2 k + 2 < d_2 (k + 1)$.

The lemma follows by induction. □

We are now in a position to prove the main theorem of this section.

Theorem 2. *Given an n -vertex triangulation $G = (V, E)$ embedded in the $n \times n$ grid with straight-line segments as edges, with $O(n^2)$ edge moves and $O(n)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G into the canonical triangulation.*

Proof. Let R represent the points of the $n \times n$ grid containing G and let P represent the universal point set. First apply Lemma 3 to G . Then, we can apply Lemma 4. The theorem follows. \square

An immediate corollary is the following.

Corollary 1. *Given two n -vertex triangulations $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ embedded in the $n \times n$ grid with straight-line segments as edges, with $O(n^2)$ edge moves and $O(n)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G_1 into G_2*

Remark: We note that with a little care, our grid size can be reduced to $3n \times 3n$ at the expense of simplicity of exposition. We chose to keep the explanations simple in order to easily convey the main ideas rather than get bogged down in details.

3 Transforming One Tree to Another

In this section, we show that $O(n)$ point and edge moves suffice to transform one tree into another and this is optimal as there are pairs of trees that require $\Omega(n)$ point and edge moves to transform one into the other.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two trees embedded in the plane on an $n \times n$ grid with $|V_1| = |V_2| = n$. Let the origin $(0, 0)$ of this grid P be the bottom left corner. During the whole sequence of moves, the location of every point move is a grid point of P . The approach is similar to that used for triangulations, but since trees are a simpler structure, the number of moves and the grid size are reduced.

Avis and Fukuda [1] showed that given any tree embedded in the plane, with at most $n - 2$ edge moves, this tree can be transformed into a canonical tree. The canonical tree they use is the star from the leftmost vertex. Moreover, the sequence of edge moves is such that each new edge added does not intersect any of the edges in the current tree. That is, if T is the current tree and the edge move consists of adding $e \notin T$ and deleting $f \in T$, then $T \cup e$ is a plane graph. We call such an edge move a *planar edge move*. Unfortunately, we cannot use this result directly, since it is proved for points in general position where no three points are collinear and we are dealing with trees embedded in the grid. However, we modify their result to account for the collinearities.

The canonical tree can no longer be the star from the leftmost point since there may be collinearities. Let p_1, p_2, \dots, p_n be the vertex set of the given tree T . Relabel the points in the following manner. Let p_1 be the leftmost, bottommost point. Label the other points p_2, \dots, p_n in sorted order counter-clockwise around p_1 so that p_1p_2 and p_1p_n are on the convex hull, and if p_1, p_i, p_j are collinear, then $i < j$. The canonical tree is the following: the edge p_1p_i is in the tree if there is no point $p_j, j \neq i$ in the interior of the segment p_1p_i . If the segment p_1p_i has

points in its interior, let p_k be the interior point closest to p_i . The segment $p_k p_i$ is in the tree. Note that essentially this builds paths of collinear vertices from p_1 . The paths of collinear vertices shall be referred to as *tentacles* of p_1 . See Figure 6 for an illustration.

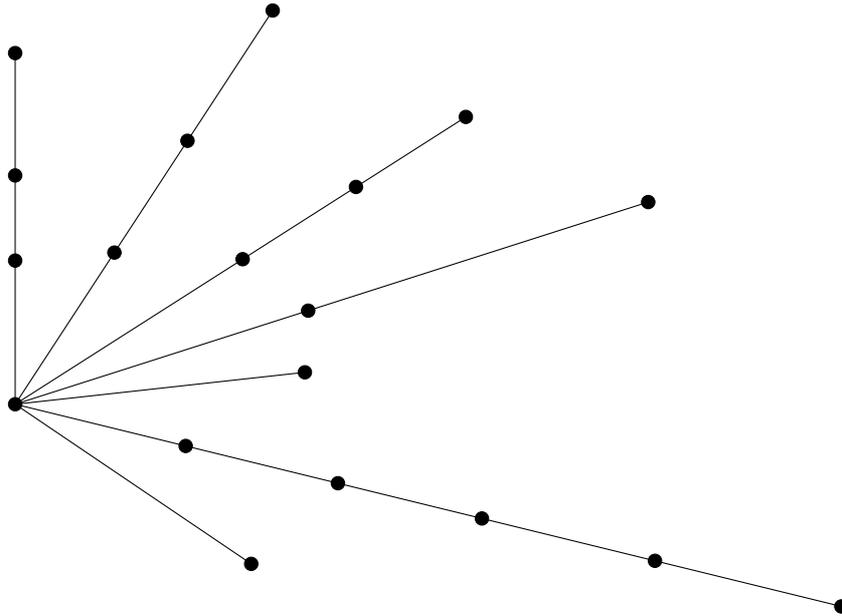


Figure 6: Canonical Tree.

Lemma 5. *A tree T with n -vertices embedded in the $n \times n$ grid can be transformed into the canonical tree with $n - 2$ edge moves. Each edge move is planar.*

Proof. Let T be the given tree embedded on the points p_1, \dots, p_n labeled as above. Call an edge $p_i p_j$ of T a *transversal edge* if the line through $p_i p_j$ does not contain p_1 . We proceed by induction on the number t of transversal edges.

Base Case: $t = 0$. In this case, T is the canonical tree.

Inductive Hypothesis: $t < k, k > 0$. With t edge moves, T can be transformed into the canonical tree.

Inductive Step: $t = k$. Avis and Fukuda [1] show that for points in general position, if T is not a star from p_1 , there always exists an edge $p_i p_j \in T$ such that no other edge of T intersects the interior of the triangle $\Delta(p_1 p_i p_j)$. We cannot apply their result directly since the points of T are not in general position. However, the same argument shows the existence of a transversal edge $p_i p_j$ such that for any point p in the interior of segment $p_i p_j$, the segment $p_1 p$ does not intersect any other transversal edge.

Now, removing $p_i p_j$ disconnects T into two components C_1 containing p_i and C_2 containing p_j . Without loss of generality, let p_1 be in C_1 . Let $p_1 = x_1, x_2, \dots, x_a = p_j$ be the vertices of T on the segment $p_1 p_j$. Since $p_1 \in C_1$ and $p_j \in C_2$, there exists a k such that $x_k \in C_1$ and $x_{k+1} \in C_2$. Add edge $x_k x_{k+1}$ to the tree. Since we have reduced the number of transversal edges with one edge move, the result follows by induction.

□

Lemma 5 gives us the freedom to move from any one tree T_1 to any other tree T_2 defined on the same point set with $2n - 4$ edge moves since we can transform one tree to the other via the canonical tree. Now, to transform a tree embedded on one point set to another tree embedded on a different point set, we need to perform some point moves. Given an n -vertex tree T embedded in the $n \times n$ grid, we show how to transform it into a path embedded on vertices $(0, i)$, $0 \leq i \leq n - 1$. Let p_1, p_2, \dots, p_n be the n points of T . Relabel these points so that they are sorted by increasing X coordinate with p_1 being the leftmost, bottommost point. If two points p_i and p_j are on the same vertical grid line, then $i < j$ if p_i is below p_j . Now Lemma 5 implies that T can be transformed to the path p_1, p_2, \dots, p_n with $2n - 4$ edge moves. We call such a path a *monotone path* (see Figure 7 for an illustration of such a path).

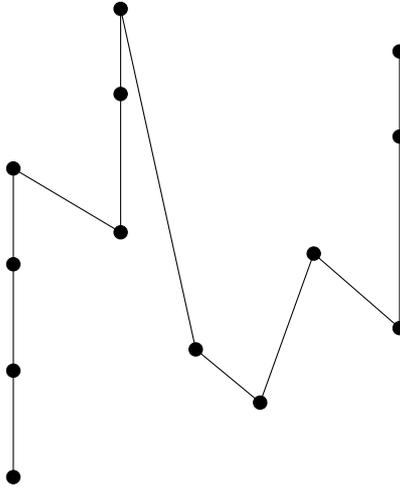


Figure 7: A monotone path.

Lemma 6. *A monotone path embedded on the $n \times n$ grid can be transformed to the canonical path embedded on vertices $(0, i)$, $0 \leq i \leq n - 1$ with n point moves.*

Proof. By definition, the half-plane to the left of the vertical line through the leftmost point is empty. Therefore, the leftmost, bottommost point can be moved to any grid point below it and to its left. Move it to $(0, 0)$. Once this point is moved, the next leftmost, bottommost point can be moved to $(0, 1)$. The lemma follows by induction. \square

Theorem 3. *Given two trees T_1 and T_2 embedded on the $n \times n$ grid, with at most $4n - 8$ edge moves and $2n$ point moves, T_1 can be converted to T_2 . All moves remain in the original grid.*

Proof. The theorem follows from the discussion above and Lemmata 6 and 5. \square

In order to show the lower bound, take an n -vertex star and an n -vertex path each embedded on n different grid points. To convert the path to a star, we need at least $n - 3$ edge moves since all vertices of the path have degree at most 2 and the star has a vertex of degree $n - 1$. Similarly, since none of the points of the star coincide with the points of the path, we need at least n point moves to get from the vertex set of the path to that of the star.

Theorem 4. *There exist pairs of trees T_1 and T_2 embedded on the $n \times n$ grid that require at least $n - 3$ edge moves and at least n point moves to transform one to the other.*

4 Transforming One Plane Graph to Another

We now show how to generalize the results from Section 2 to plane graphs. Given two plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ embedded in the $n \times n$ grid with $|V_1| = |V_2| = n$ and $|E_1| = |E_2| = m$, we show how to transform G_1 into G_2 . We will assume that both graphs are connected. The obvious approach is to add *dummy edges* to both graphs until they are triangulations. Then, apply the previous result and ignore the moves that concerning dummy edges. Although this basic approach works, we need to address a few details along the way.

We will show how to transform G_1 into a canonical form. The problem is that since G_1 is not a triangulation, we need to specify precisely what the canonical form is. Recall the canonical form for triangulations and label its vertices in the following way. Let p_1 and p_2 be the left and right corners of the outerface and let p_3 be the apex. Label the vertices p_4, \dots, p_n in descending order on the spine from p_3 . Label the edges adjacent to p_1 by e_0, \dots, e_{n-2} in clockwise order around p_1 with $e_0 = p_1p_3$ and $e_{n-2} = p_1p_2$. Label all the edges adjacent to p_2 except edge p_1p_2 and p_2p_3 by e_{n-1}, \dots, e_{2n-5} , in counter-clockwise order with $e_{n-1} = p_2p_4$ and $e_{2n-5} = p_2p_n$.

Now, the value of m determines the shape of the canonical graph. Since G_1 is connected and planar, $n - 1 \leq m \leq 3n - 6$. If $m = n - 1$, then the canonical graph is a tree formed by the path from p_3 to p_n along with the edges p_1p_3 and p_2p_3 . If $m > n - 1$, let $k = m - n + 1$. Augment the canonical tree with the edges e_1, \dots, e_k .

The first step is to triangulate G_1 . Bicolor the edges red and blue so that the original m edges are red and all additional edges are blue. Next, we show how to move three of the vertices of G_1 to the coordinates $(-2n, -1)$, $(3n, -1)$ and $(\lceil n/2 \rceil, 3n)$.

Lemma 7. *Given a near-triangulation G embedded in the $n \times n$ grid, with $O(n)$ moves and at most 8 point moves, we can transform it into a near-triangulation whose outerface has three vertices at coordinates $(-2n, -1)$, $(3n, -1)$ and $(\lceil n/2 \rceil, 3n)$. All other vertices of G have coordinate values between 0 and n (i.e. they are in the original $n \times n$ grid).*

Proof. Similar to the proof of Lemma 3 since the outerface is a convex polygon so one of the vertices of the polygon can be moved vertically and one horizontally. \square

By applying Lemma 7 to G_1 , we convert it to a graph whose convex hull consists of 3 vertices. Now, we can add more blue edges such that G_1 is a triangulation and no longer necessarily a near-triangulation. This permits us to apply Theorem 2, which results in a canonical triangulation with m red edges and the remaining are blue. The red edges are not necessarily in canonical

position but at most m edge moves allow one to reshuffle the edges into correct position. Therefore, we have the following:

Theorem 5. *Given an n -vertex plane graph $G = (V, E)$ embedded in the $n \times n$ grid with straight-line segments as edges, with $O(n^2)$ edge moves and $O(n)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G into the canonical plane graph.*

Corollary 2. *Given two n -vertex plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ embedded in the $n \times n$ grid each having m edges, with $O(n^2)$ edge moves and $O(n)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G_1 into G_2*

One aspect of this approach which may be unsatisfactory is that throughout the sequence, the graph may become disconnected eventhough we start with a connected graph. This begs the question: is there a way to guarantee that in converting one plane graph into another, we remain connected throughout the whole sequence? We answer this in the affirmative with a slight increase in the number of edge moves but the total number of edge moves remains quadratic. Note that point moves do not change the connectivity of a graph. Therefore, we solely need to concentrate on edge moves. The main tool we used for edge moves in triangulations is Lemma 1. The key idea is to maintain a connected spanning red graph after every edge flip. We prove the following.

Lemma 8. *Let G be an n -vertex near-triangulation. Let a subset of the edges of G be colored red such that the graph induced by the red edges is connected and spanning. The remaining edges of G are colored blue. Let e be an edge of G to be flipped. With at most 1 edge move, we can flip e such that the graph induced by the red edges remains connected and spanning, after each of the edge move and edge flip.*

Proof. Let R be the graph induced by the m red edges. We need to show that we can flip an edge e of G such that R remains connected after the flip. Let e be the edge to be flipped. If e is blue, then flipping e does not affect the connectivity of the graph induced on the red edges. If e is red, then the only way that the connectivity of R is affected is if e is a cut edge[§] of R .

Since e is in G , e is adjacent to at least one triangular face of G . Let a, b, c with $e = ab$ be the three vertices defining this face. The edges bc and ac cannot both be red since this would contradict the fact that e is a cut edge. Since e is a cut edge, the deletion of e from R disconnects the graph into two components with a and b going to different components. Without loss of generality, assume that b and c are in different components. Then performing an edge move in the red graph from $e = ab$ to bc , we have a new set of m red edges that form a connected and spanning subgraph of G . Essentially, this amounts to coloring e blue and bc red. Now, since e is blue, we can flip e without affecting the connectivity of R .

Therefore, after one edge move, we can perform the flip. The lemma follows. □

Since Lemma 1 uses edge flips in a near-triangulation, and these are the only edge moves we use, we conclude with the following.

[§]A cut edge is an edge whose deletion disconnects a graph

Corollary 3. *Given two connected n -vertex plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ embedded in the $n \times n$ grid each having m edges, with $O(n^2)$ edge moves and $O(n)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G_1 into G_2 while remaining connected throughout the sequence of moves.*

5 Labeled Transformations

In this section, we address the same problems in the labeled setting, that is we impose an initial mapping prior to the transformation. Specifically, given two plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we define a mapping $\phi : V_1 \rightarrow V_2$. Now perform a sequence of edge and point moves that transforms G_1 into a graph $G_3 = (V_3, E_3)$ that is isomorphic to G_2 . There is a mapping $\delta : V_1 \rightarrow V_3$. In the unlabeled case, we simply want G_3 to be isomorphic to G_2 . In the labeled case, in addition, we want for every vertex $x \in V_1$, that $\phi(x) = \delta(x)$.

5.1 Labeled Trees

In the case of trees, the labeled setting is fairly straightforward. We are given two trees T_1 and T_2 as well as a mapping ϕ . Lemmata 5 and 6 imply that both T_1 and T_2 can be converted into a canonical tree, which is a path on one row of the grid. Given the two canonical paths, the only problem is the labeling. We show a simple way to permute the labels.

Assume without loss of generality that the path is on the X axis with vertices at points $(0, 0), (2, 0), (4, 0), \dots, (2n - 2, 0)$. Move $(0, 0)$ to $(0, 1)$. With $n - 1$ edge moves, convert the path to a star rooted at $(0, 1)$. Now, if we wish to permute the location of the vertex at $(2i, 0)$ with that of $(2j, 0)$ simply move the vertex $(2j, 0)$ to $(2i + 1, 0)$, vertex $(2i, 0)$ to $(2j, 0)$ and $(2i + 1, 0)$ to $(2i, 0)$. In this way, we can sort the labels of the vertices with $O(n)$ edge and point moves.

Theorem 6. *Given two trees T_1 and T_2 embedded on the $n \times n$ grid, and a mapping ϕ of the vertices of T_1 to the vertices of T_2 , with $O(n)$ point and edge moves, T_1 can be converted to T_2 respecting the mapping.*

Proof. Follows from the discussion above and Theorem 3. □

5.2 Labeled Triangulations and Plane Graphs

In the labeled setting for triangulations and plane graphs, we simply need to show that given any labeling of the vertices of a canonical triangulation, we move the vertices to get a different labeling of the vertices. The idea is similar to the previous section. We will show how to permute the labels on the spine and how to move a label from the spine to the outerface.

First we show how to permute the vertices on the spine. In order to do this, we simply need to show how to transpose the position of two vertices. We first modify the spine of the canonical triangulation such that between every vertex there is at least one grid point. We also make sure that the triangle of the outerface has base angles less than $\pi/4$. This will ensure that there is a grid point in every triangle by Observation 2 and make the following moves possible.

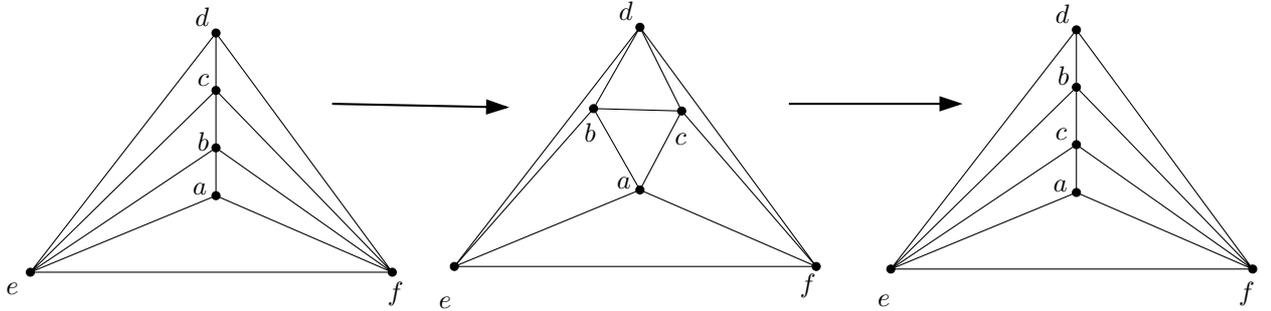


Figure 8: Permuting the Spine of the Canonical Triangulation.

Let a, b, c, d be four consecutive vertices on the spine and we wish to permute b and c . See Figure 8. With one point move, move c one grid point to the right. Now e, b, c, d form a convex quadrilateral, so flip edge ec for bd . Next, move b one grid point to the left and flip edge fb for ca . Now move b to the same height as c . This gives us the triangulation in the middle of Figure 8. Reverse the steps to change the order of b and c on the spine. So with $O(1)$ point moves and edge moves, we can permute the position of two consecutive vertices on the spine.

Lemma 9. *With $O(n^2)$ point and edge moves, we can go from any permutation of the spine to any other permutation.*

Proof. Follows from the discussion above and the fact that $O(n^2)$ transpositions are sufficient to sort any permutation of n numbers. \square

Next, it is easy to see that a vertex d can be moved from the spine to the outerface with one move, as illustrated in Figure 9. Then, with $O(n)$ edge moves and two point moves, the vertex e that was in the outerface, can be moved to the bottom of the spine.

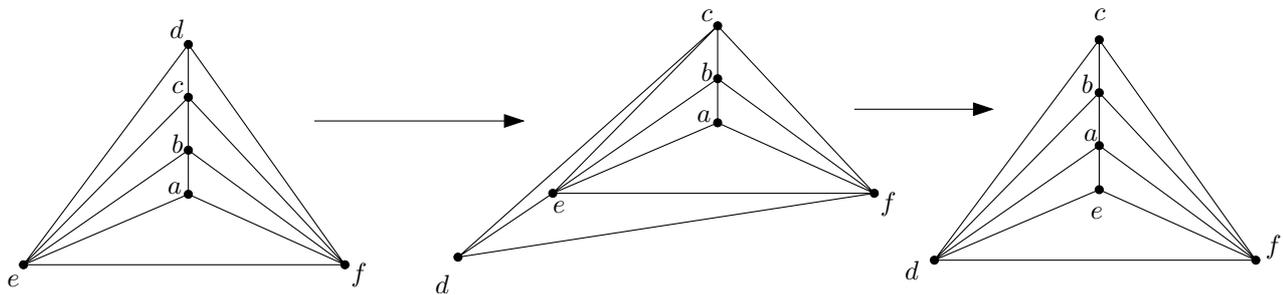


Figure 9: Moving a vertex from the spine to the outerface.

This implies that all of the results on moving plane graphs translate to the labeled setting at the additional cost of $O(n^2)$ point and edge moves.

Theorem 7. *Given two n -vertex plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ embedded in the $n \times n$ grid each having m edges, and a mapping ϕ of the vertices of G_1 to the vertices of G_2 , with $O(n^2)$ edge moves and $O(n^2)$ point moves (all point moves stay within the grid $[(-2n, -2n), (3n, 3n)]$), we can transform G_1 into G_2 while respecting the given mapping and remaining connected throughout the sequence of moves.*

6 Conclusion

We have shown how point and edge moves suffice to transform any plane graph to any other plane graph embedded in the plane with a finite number of moves on a small grid. As for further research, it would be interesting to either reduce the number moves used or find matching lower bounds in the general setting.

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