Recent developments on the crossing number of K_n

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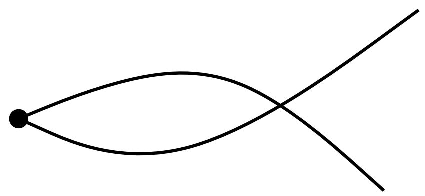

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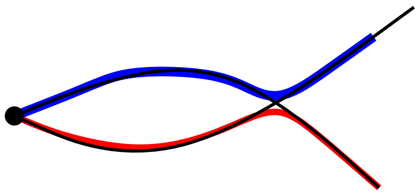

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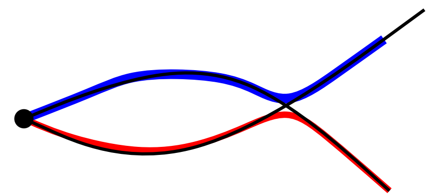

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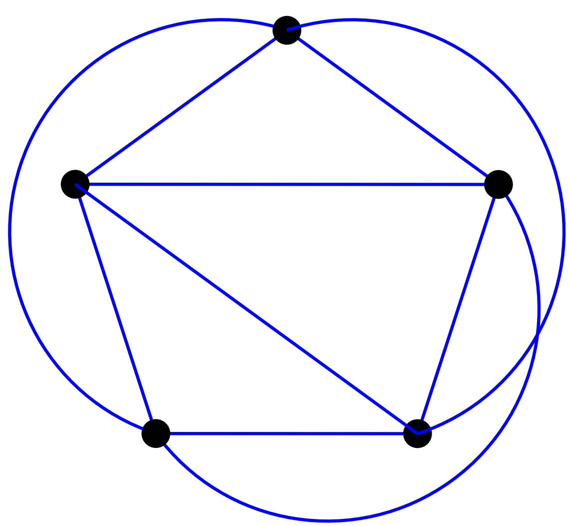

$$\mathrm{cr}(K_5) = 1$$

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  * If we add a single edge e to a plane graph G, computing $\mathrm{cr}(G \cup \{e\})$ is also NP-hard. [Cabello-Mohar, 2010]

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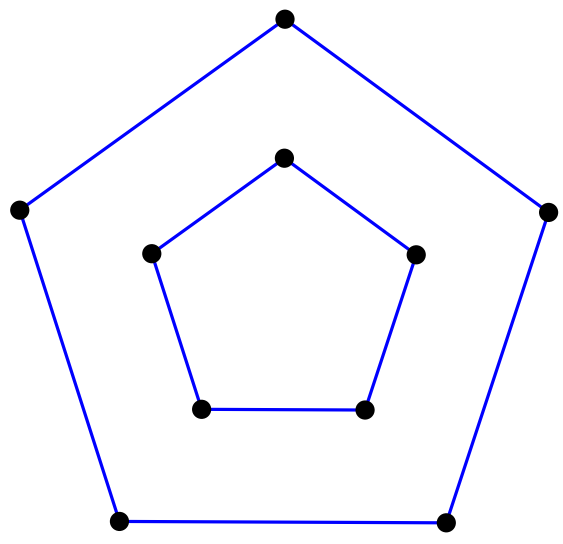

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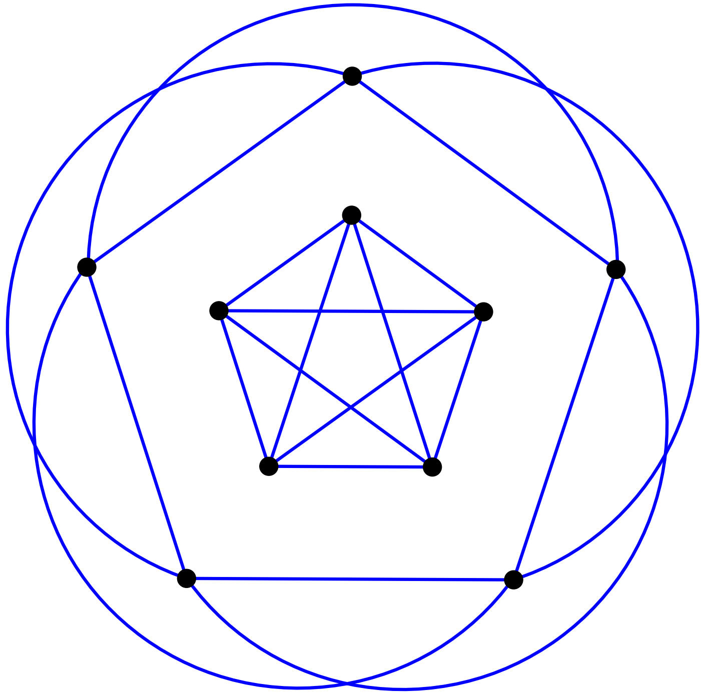

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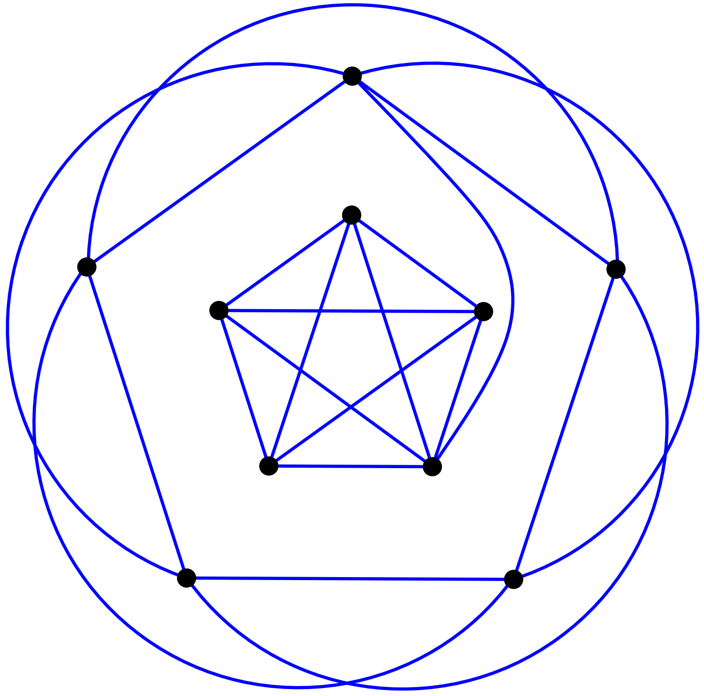

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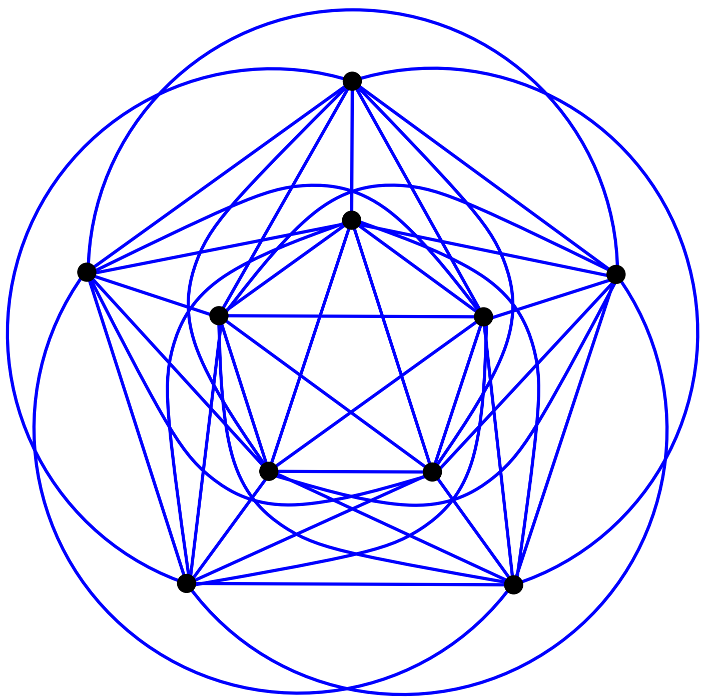

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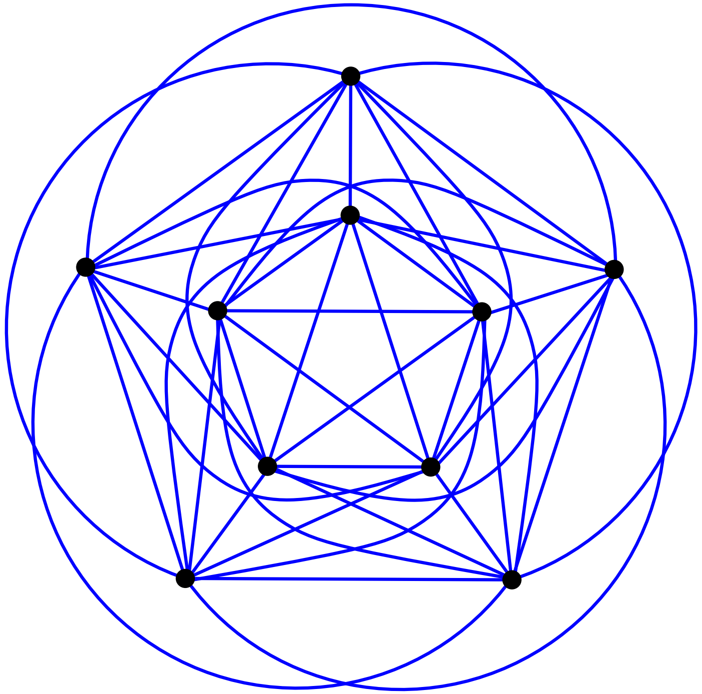

The number of crossings in these drawings is

$$Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

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* Some known results for small n:
  - $\Diamond$ $\mathrm{cr}(K_n) = Z(n)$ si $n \leq 10$ [Guy, 1971]
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$*$ This was the situation, till a new tool was borrowed from the rectilinear case.

Rectilinear crossing number

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  $\square(S)$: number of convex quadrilaterals in S.

$$\boxed{\overline{\mathrm{cr}}(K_n) = \min_{|S|=n} \square(S)}$$

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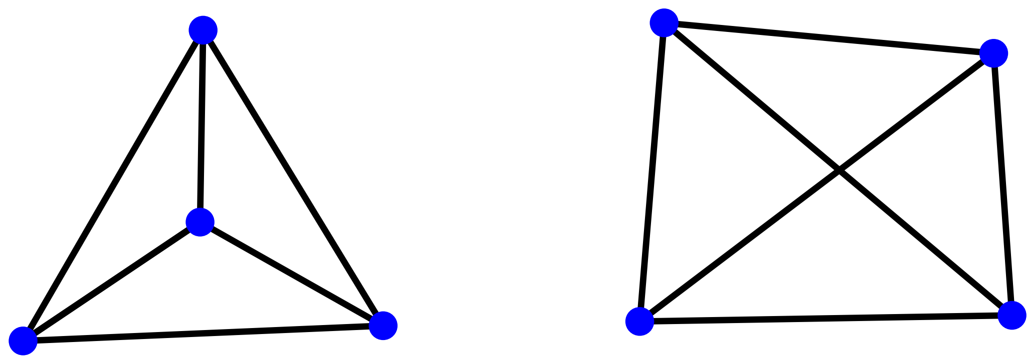

Rectilinear crossing number

* Until 2004, the status of the rectilinear problem was similar to that of the general case:
  * known for $n \leq 10$ (case analysis).
    $\overline{\mathrm{cr}}(K_{10}) = 62$ [Brodsky-Durocher-Gethner, 2001]
  * upper bound: no conjecture for an optimal construction.
  * lower bound: $\overline{\mathrm{cr}}(K_n) \geq 0.3001\binom{n}{4}$
    $(Z(n) = 0.375\binom{n}{4} + O(n^3))$

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- 2004: Ábrego - Fernández-Merchant, Lovász-Vesztergombi-Wagner-Welzl

  Relation between $\square(S)$ and the number of j-edges of S.

j-edges

* Let S be a set of n points in the plane in general position. Given $p, q \in S$, we say that pq is an (oriented) j-edge if there are j points of S in the right halpf-plane defined by pq.

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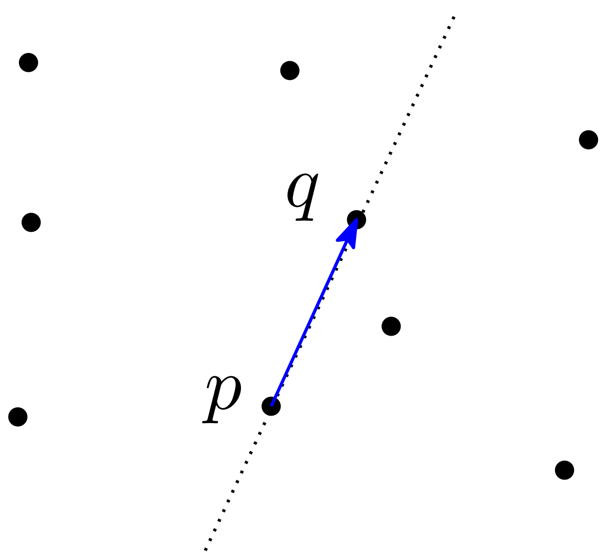

3-edge

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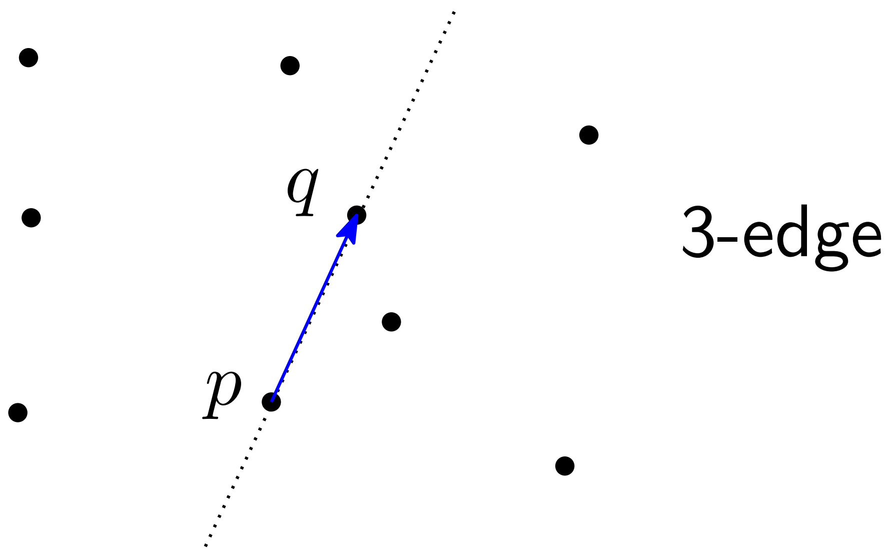

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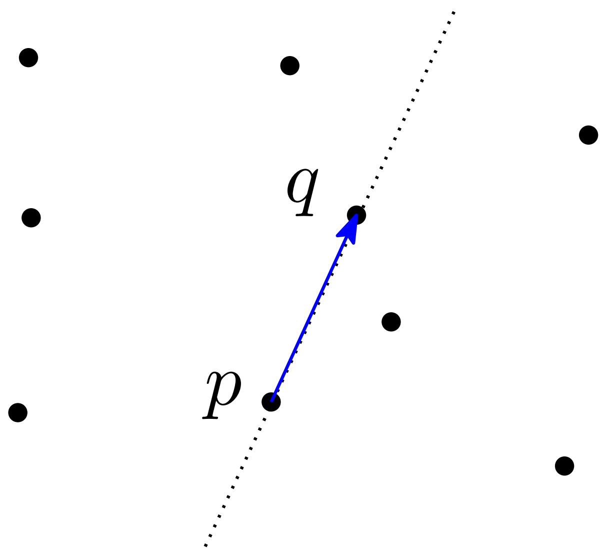


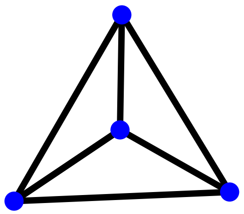
3-edge

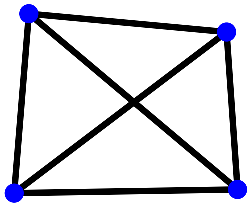
* $e_j(S)$:= # j-edges of S.

* If pq is a j-edge, then qp is a $n-j-2$-edge.
  It is also possible to work with unoriented j-edges.

j-edges and convex quadrilaterals (crossings)

$$* \quad \triangle(S) + \square(S) = \binom{n}{4} \qquad (1)$$





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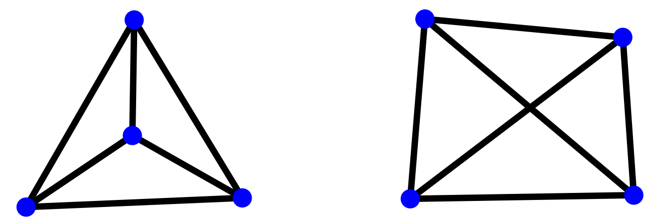
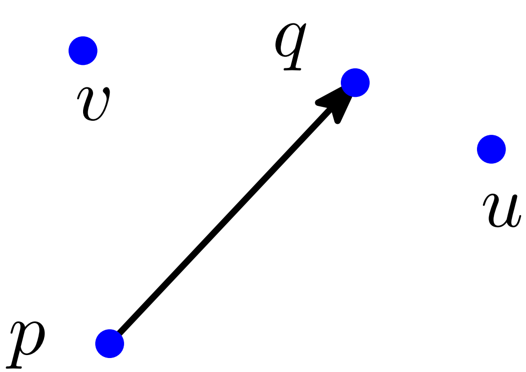


* Another relation: double counting of 4-tuples $\{p, q, u, v\}$ where the ordered pair p, q leaves u to the right and v to the left.



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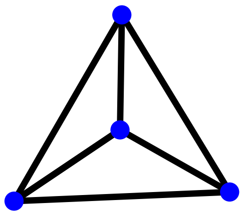

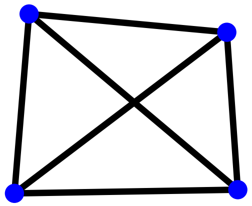

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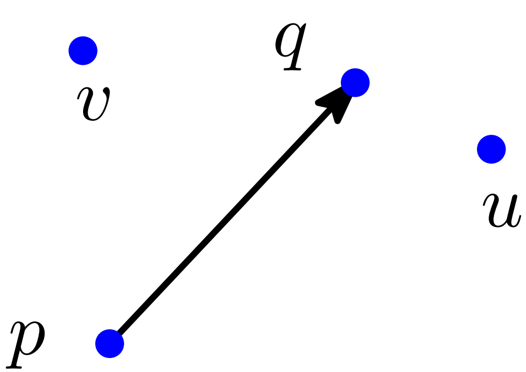

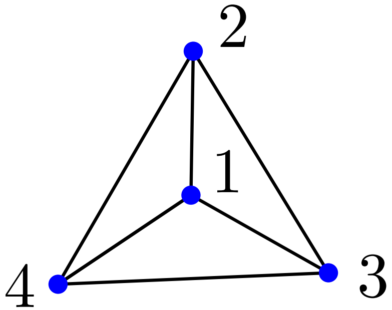

six 4-tuples

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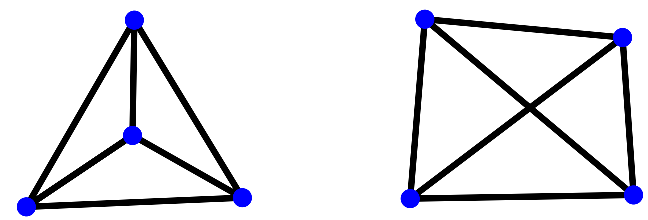

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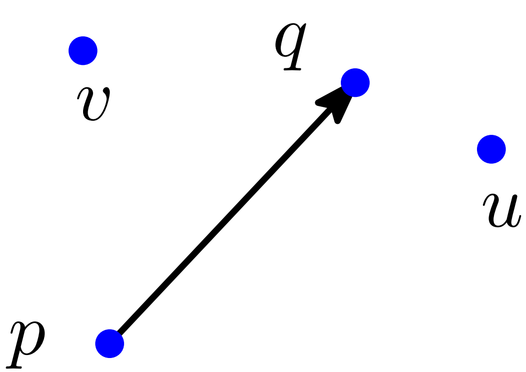

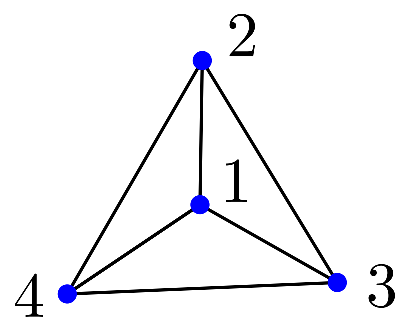

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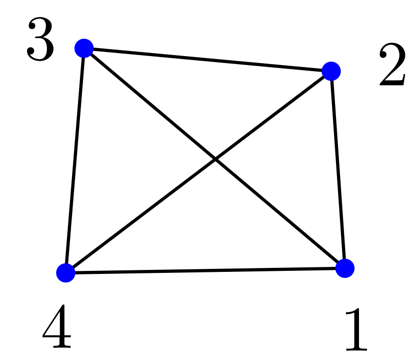

four 4-tuples

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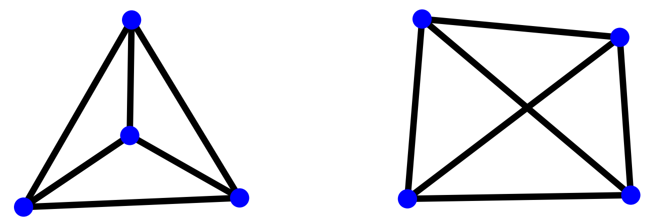
$*$ $\triangle(S) + \square(S) = \binom{n}{4}$ (1)



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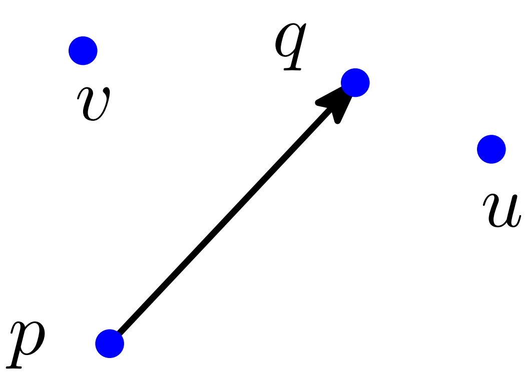

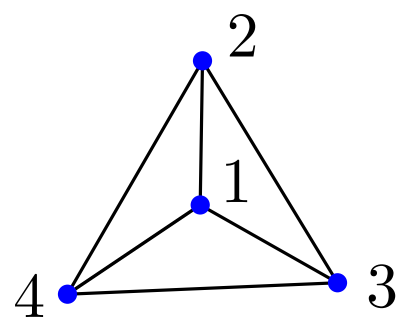

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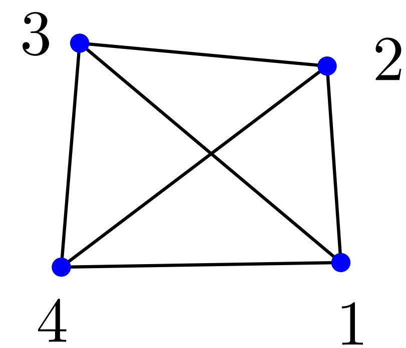

four 4-tuples

$*$ $6\,\triangle(S) + 4\,\square(S) = \sum_{j=0}^{n-2} j(n-j-2)\, e_j(S)$ (2)

j-edges and convex quadrilaterals (crossings)

* From this equation (and the relations $e_j = e_{n-j-2}$ and $\sum_{j=0}^{n-2} e_j = n(n-1)$) we get

$$\square(S) = \sum_{j < \frac{n-2}{2}} \left(\frac{n-2}{2} - j \right)^2 e_j(S) - \frac{3}{4}\binom{n}{3}$$

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* And considering $E_{\leq k}(S) = \sum_{j=0}^{k} e_j(S)$

$$\square(S) = \underbrace{\sum_{k<\frac{n-2}{2}} (n-2k-3)\, E_{\leq k}(S)}_{\text{leading term}} - \frac{3}{4}\binom{n}{3} + O(n^3)$$

Lower bounds for $\overline{\mathrm{cr}}(K_n)$

* [AF - LVWW, 2004] $E_{\leq k}(S) \geq 3\binom{k+2}{2}$

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* LVWW use an improved bound for $E_{\leq k}$ (for k close to $n/2$), to show that

$$\overline{\mathrm{cr}}(K_n) \geq 0.37501\binom{n}{4}$$

Lower bounds for $\overline{\mathrm{cr}}(K_n)$

$*$ 2004 – 2011. Series of improvements on the lower bound for $E_{\leq k}(S)$.

[Balogh-Salazar], [Aichholzer-García-Orden-R.], [Ábrego-Cetina-Fernández-Leaños-Salazar].

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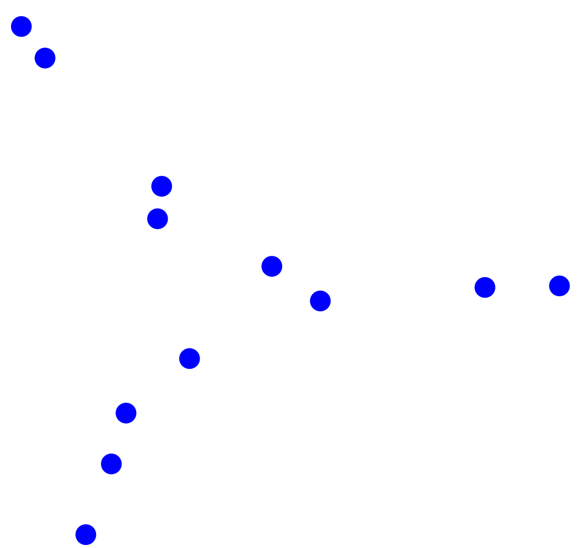

$n = 12$

Optimal set

(153 convex quads)

General (topological) drawings

* BIRS - Crossing numbers turn useful. (August 2011)

  If in the formula

$$\square(S) = \sum_{k<\frac{n-2}{2}} (n-2k-3)\, E_{\leq k}(S) - \frac{3}{4}\binom{n}{3} + c_n$$

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$$\sum_{k < \frac{n-2}{2}} (n-2k-3)\, 3\binom{k+2}{2} - \frac{3}{4}\binom{n}{3} + c_n = \boxed{\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor}$$

$$||$$

$$Z(n)$$

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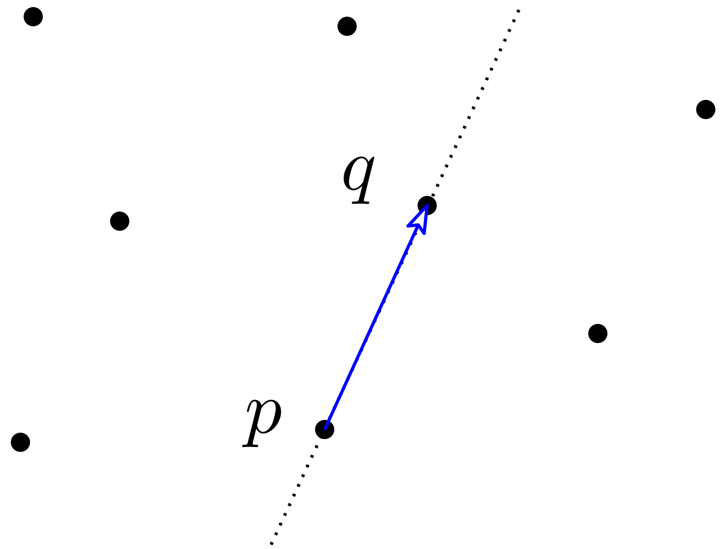
$$\sum_{k<\frac{n-2}{2}} (n-2k-3)\, 3\binom{k+2}{2} - \frac{3}{4}\binom{n}{3} + c_n = \underbrace{\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor \left\lfloor\frac{n-2}{2}\right\rfloor \left\lfloor\frac{n-3}{2}\right\rfloor}_{\parallel \atop Z(n)}$$

* Is that a coincidence?

j-edges in topological drawings



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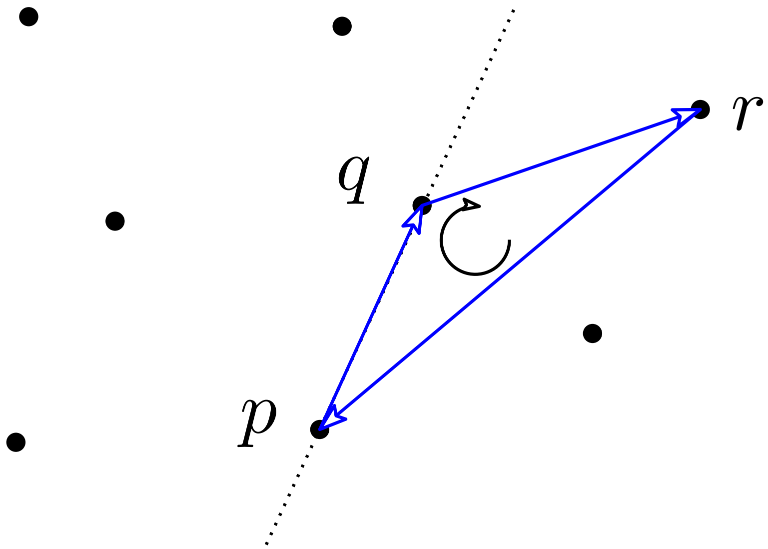

Consider the triangles!

$\sigma(pqr) = +$

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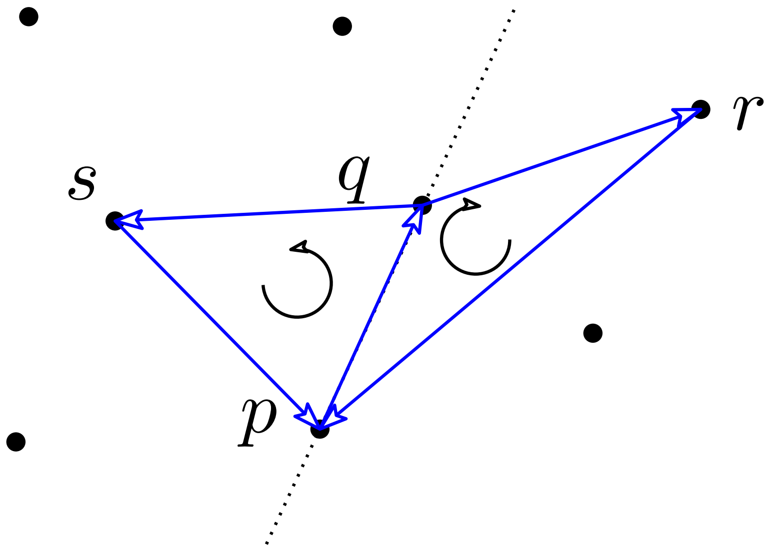

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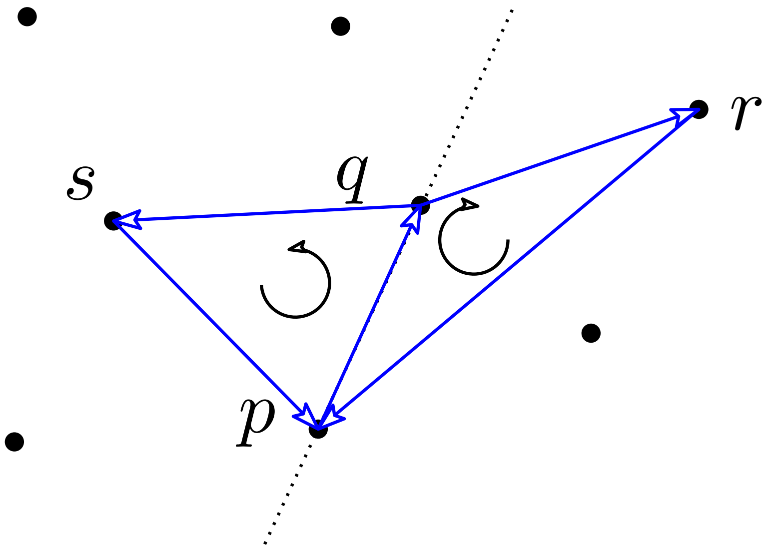

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  We say that r is to the right of pq if pqr is oriented clockwise.

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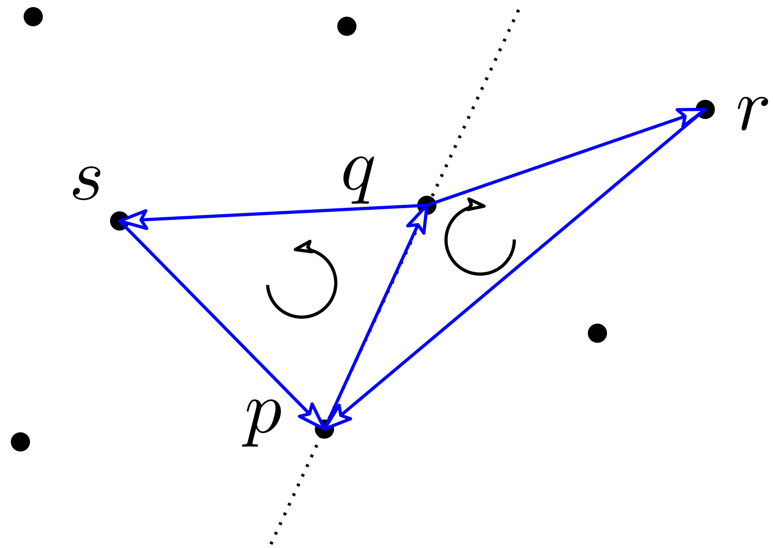

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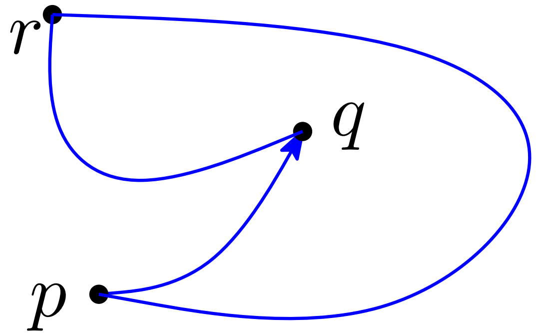
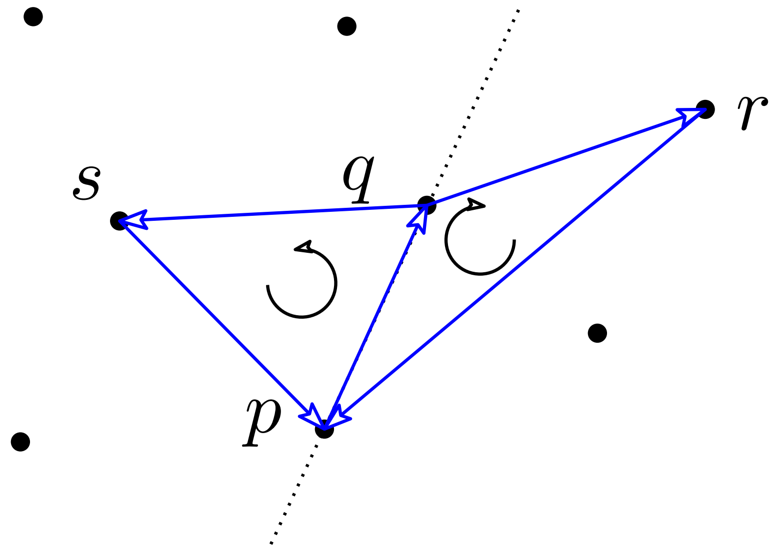


j-edges in topological drawings



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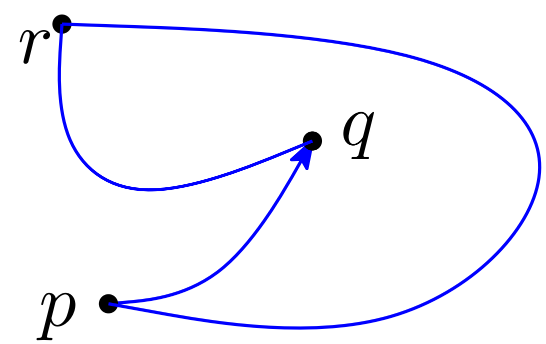

* And now we can define j-edges exactly as before.

j-edges and crossings (in topological drawings)

* Now we need to generalize the relation

$$6\,\triangle\!\!\!\!\cdot\;(S) + 4\,\square(S) = \sum_{j=0}^{n-2} j(n-j-2)\,e_j(S)$$

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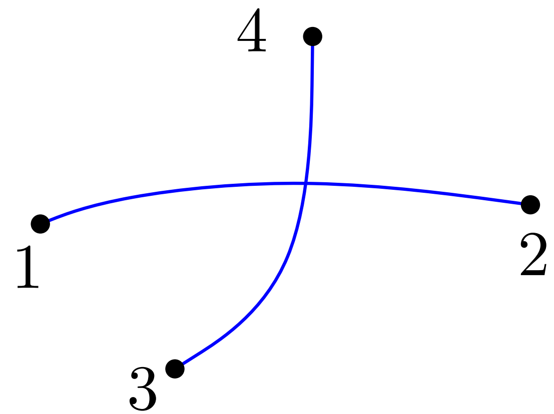

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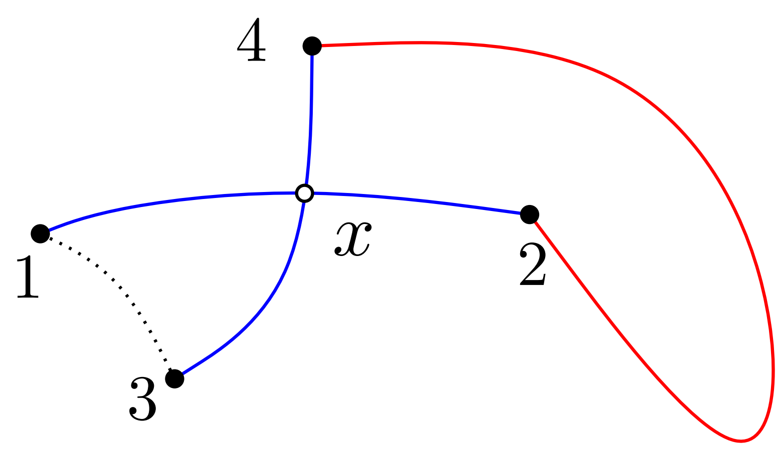
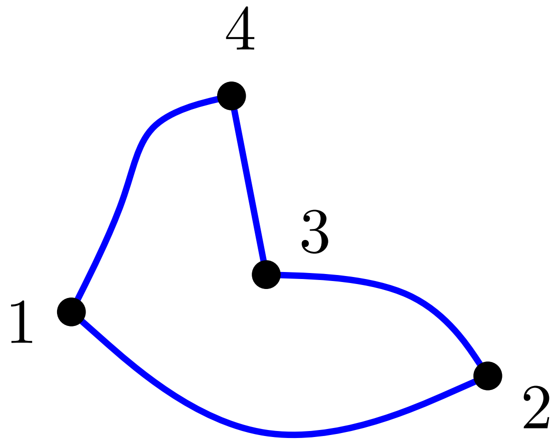
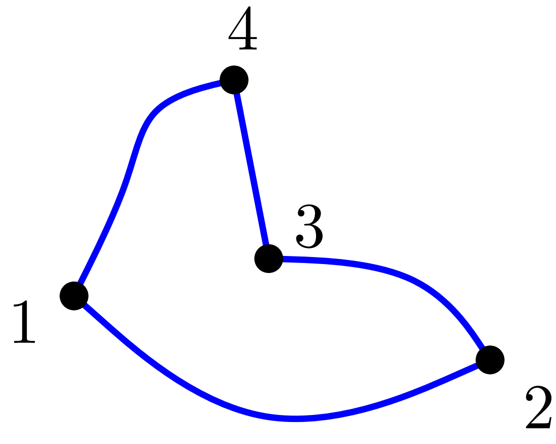
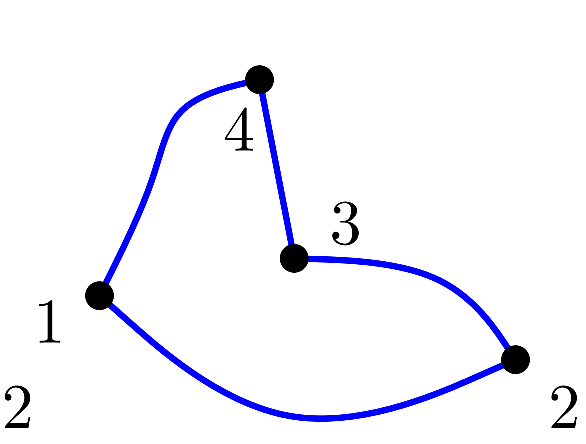


j-edges and crossings (in topological drawings)

* There are three "different" drawings of K_4.







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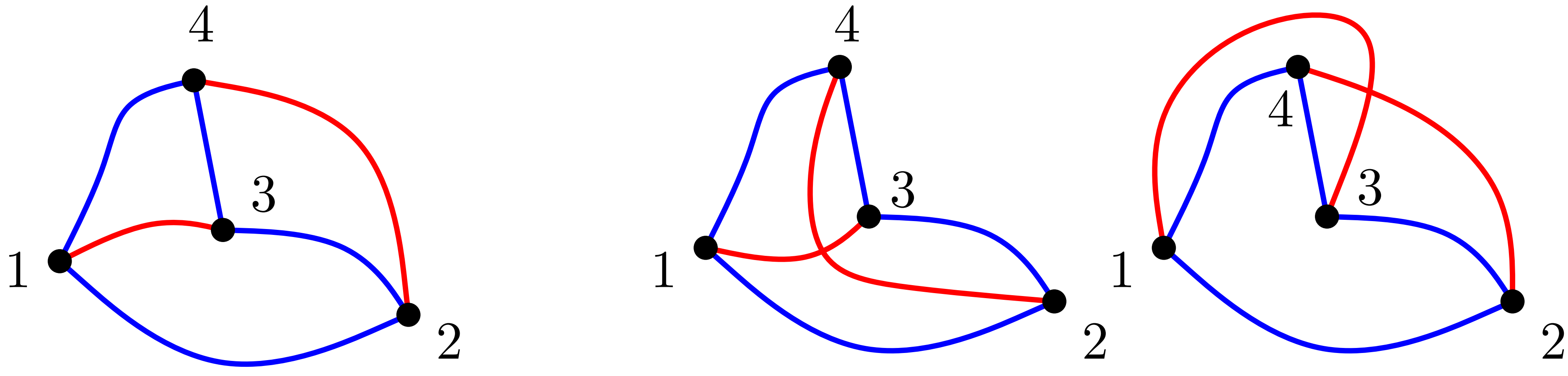

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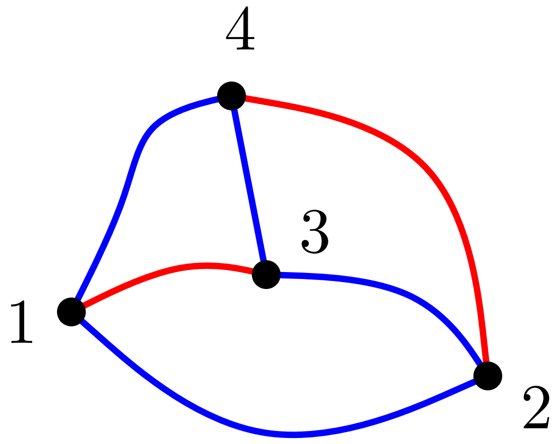

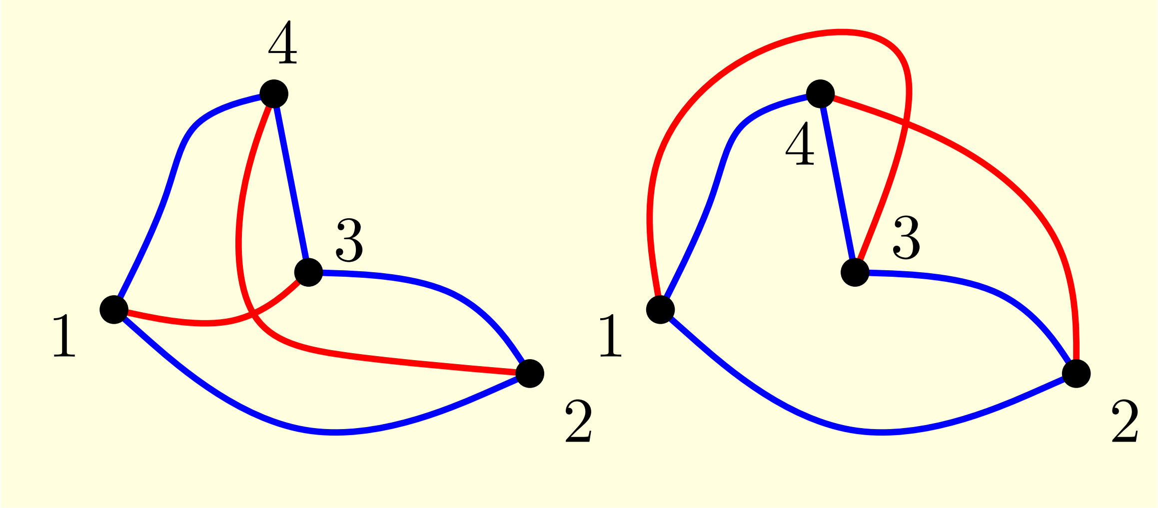

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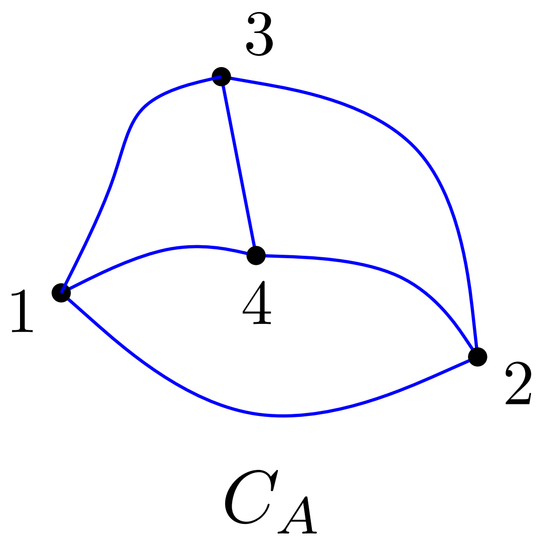

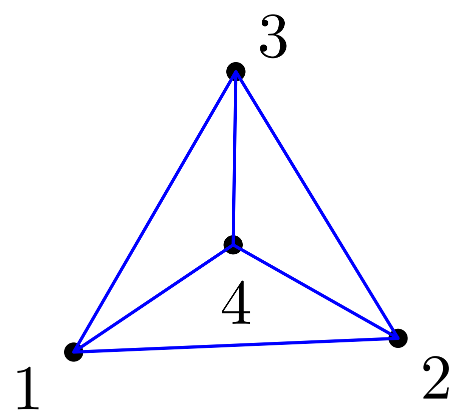

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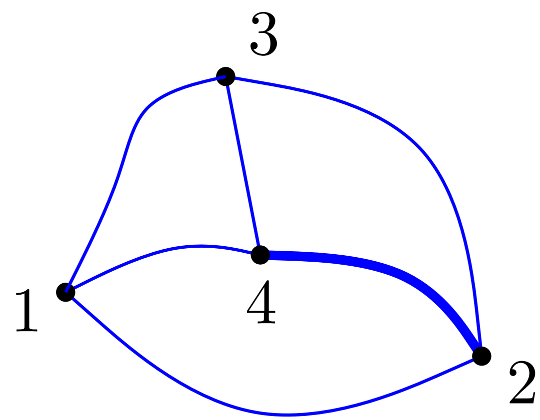
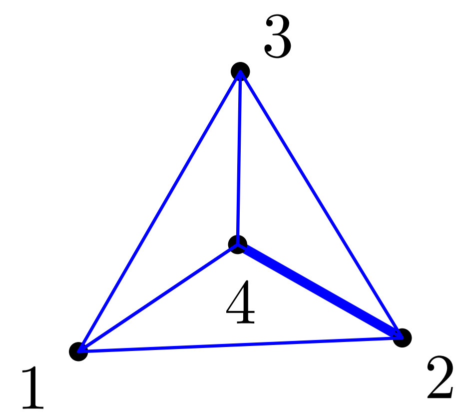


C_A



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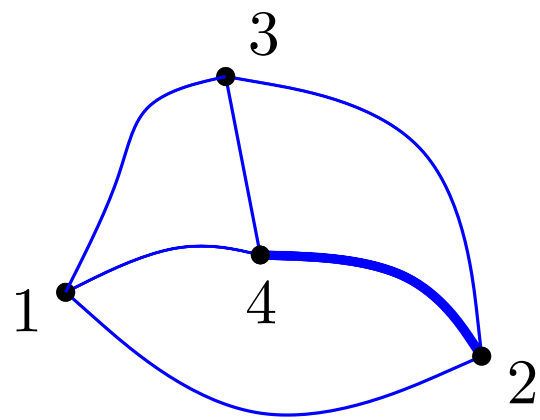
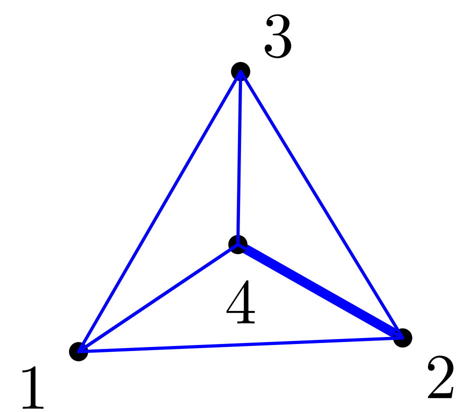
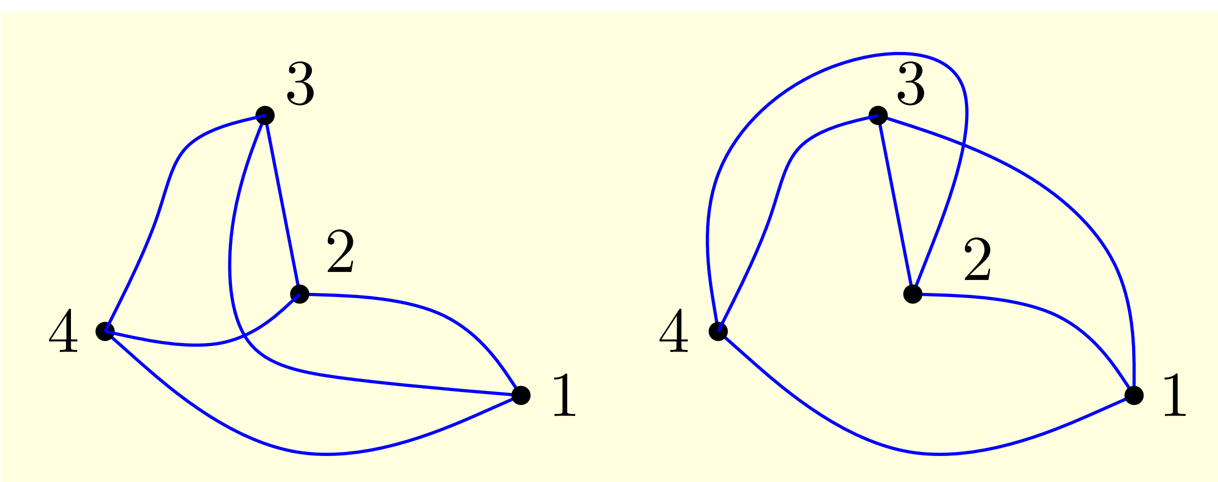
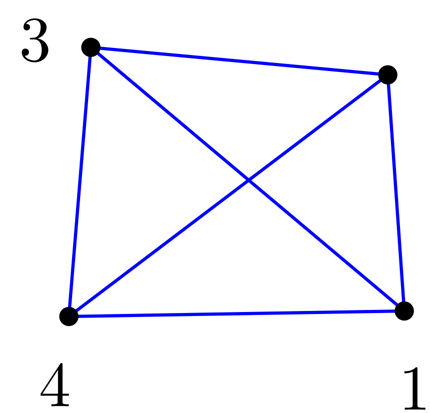




C_A





C_B

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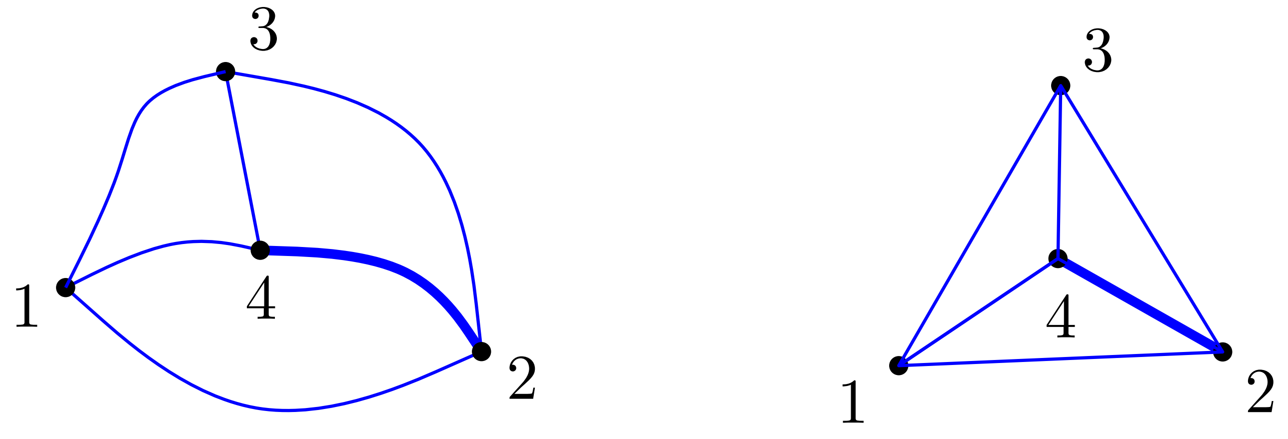
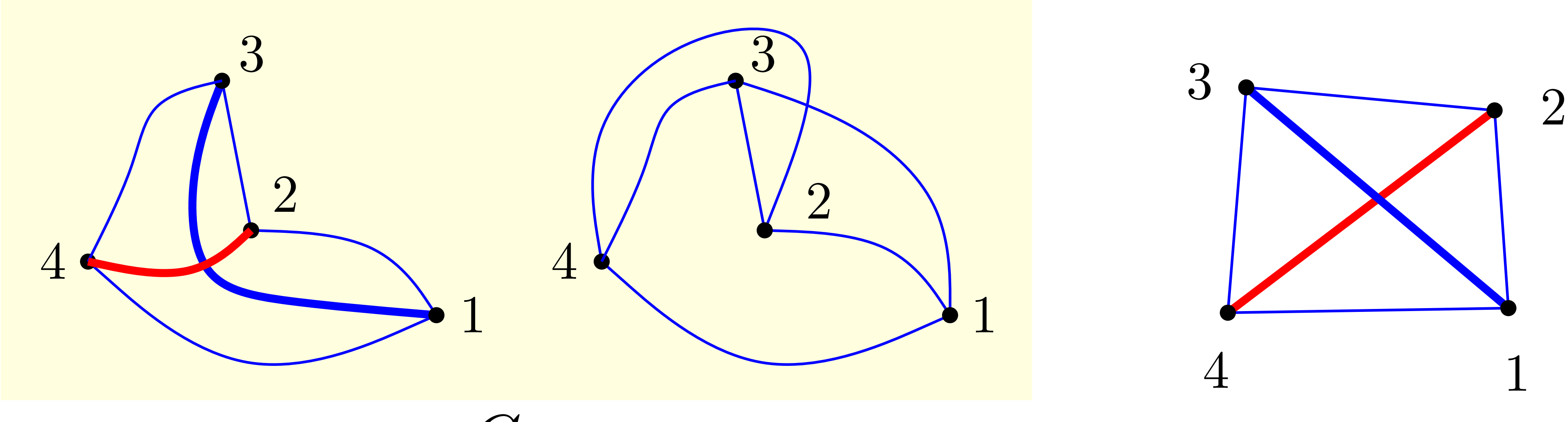


C_A



C_B

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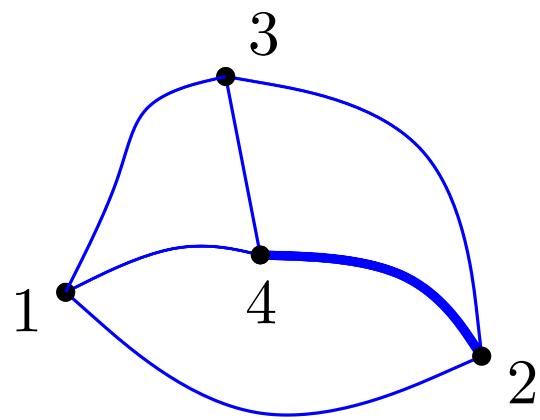
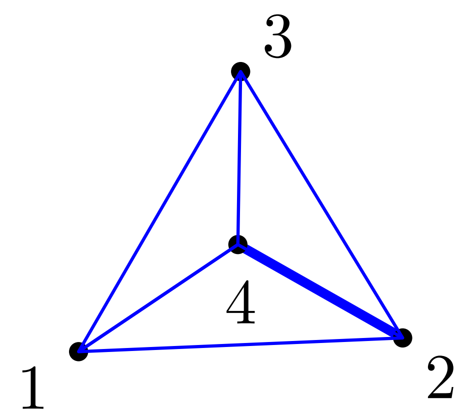
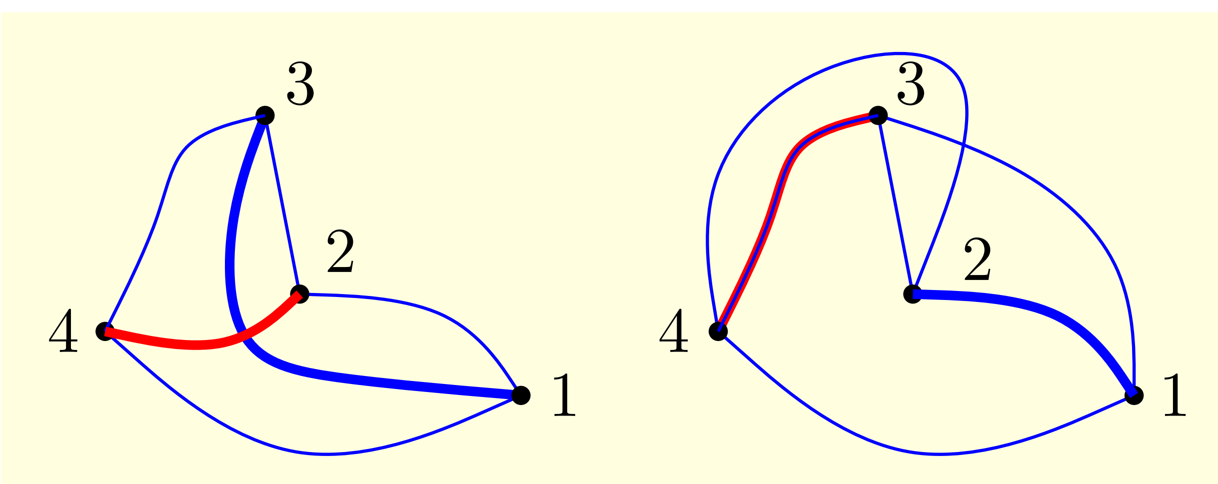
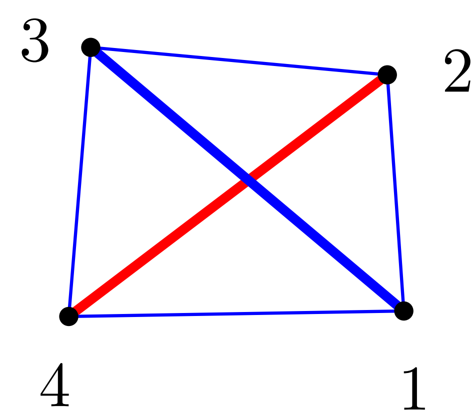




C_A



C_B



j-edges and crossings (in topological drawings)

* So we have:

1. $|C_B| = \mathrm{cr}(D)$.
2. $|C_A| + |C_B| = \binom{n}{4}$.
3. $6|C_A| + 4|C_B| = \sum_{j=0}^{n-2} j(n-j-2)\, e_j(D)$.

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$$\mathrm{cr}(D) = \sum_{j<\frac{n-2}{2}} \left(\frac{n-2}{2} - j\right)^2 e_j(D) - \frac{3}{4}\binom{n}{3}.$$

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* Finally, using $(\leq k)$-edges,

$$\mathrm{cr}(D) = \sum_{k<\frac{n-2}{2}} (n-2k-3)\, E_{\leq k}(D) - \frac{3}{4}\binom{n}{3} + c_n$$

j-edges and crossings

* If we could prove $E_{\leq k}(D) \geq 3\binom{k+2}{2}$, we would have $\mathrm{cr}(K_n) \geq Z(n)$.

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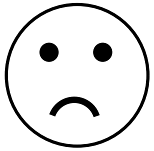
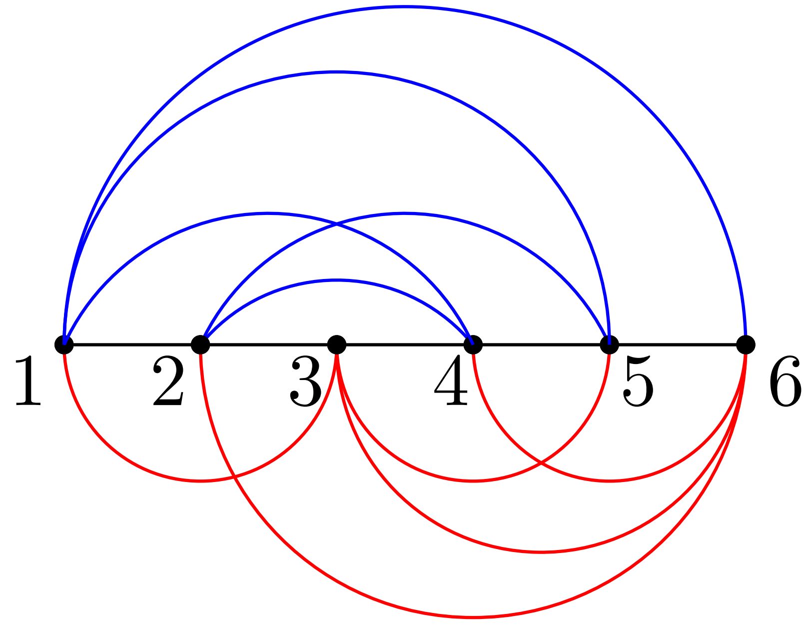
It's not true 

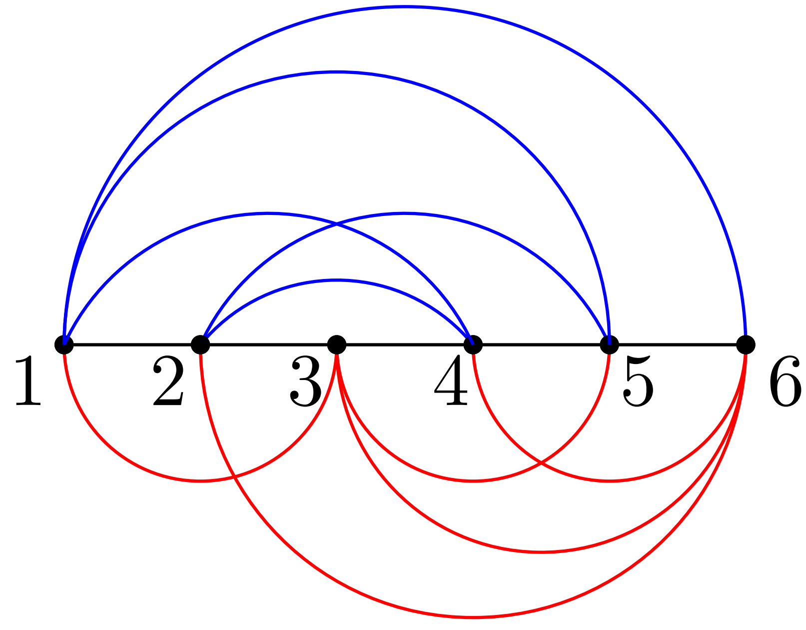
- First try: is previous lower bound for $E_{\leq k}(D)$ true for any interesting family of drawings of K_n?

2-page drawings



K_6 in two pages

2-page drawings



K_6 in two pages

crossing-free Hamiltonian cycle

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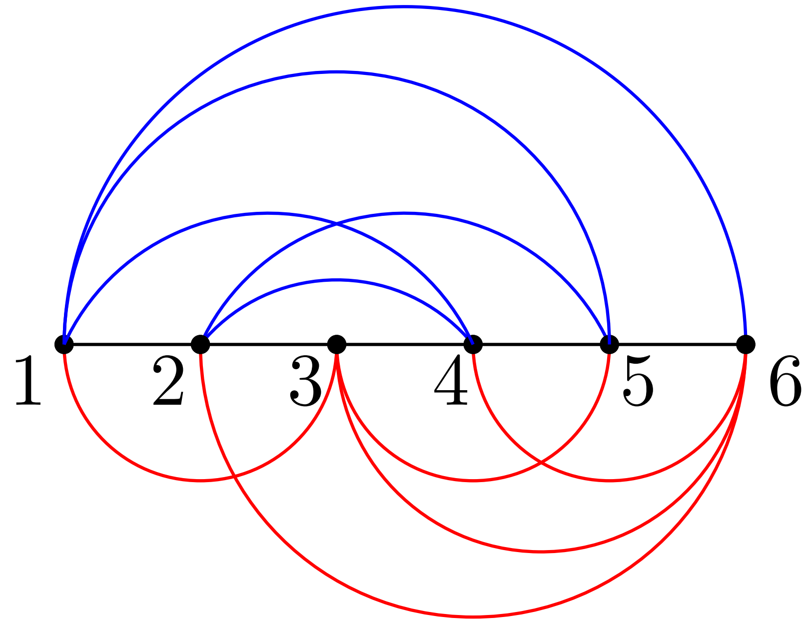

K_6 in two pages

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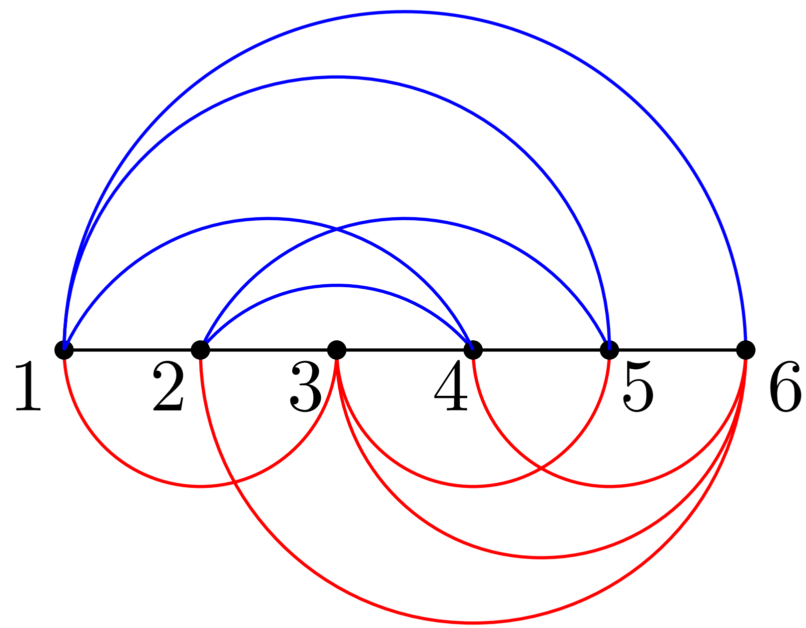
* $\nu_2(G) :=$ minimum number of crossings in any 2-page drawing of G.

2-page drawings



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crossing-free Hamiltonian cycle

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* $\nu_2(K_n) = Z(n)$

  [Ábrego, Aichholzer, Fernández-Merchant, R., Salazar, 2012]

2-page drawings

* Even for 2-page drawings,
  it is not true that

$$E_{\leq k} \geq 3\binom{k+2}{2}.$$

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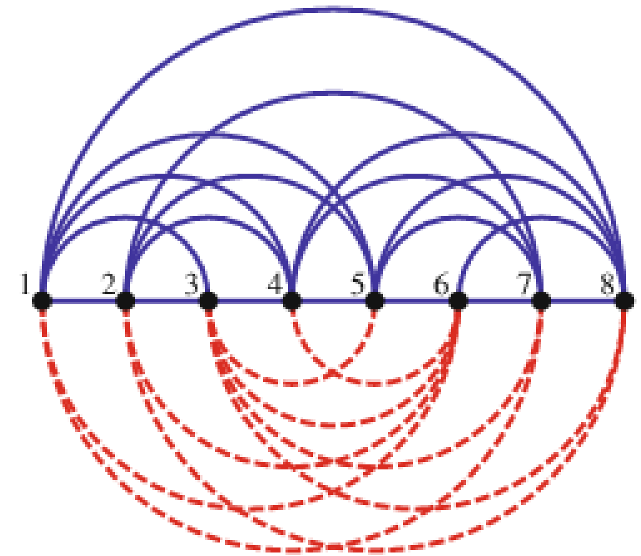

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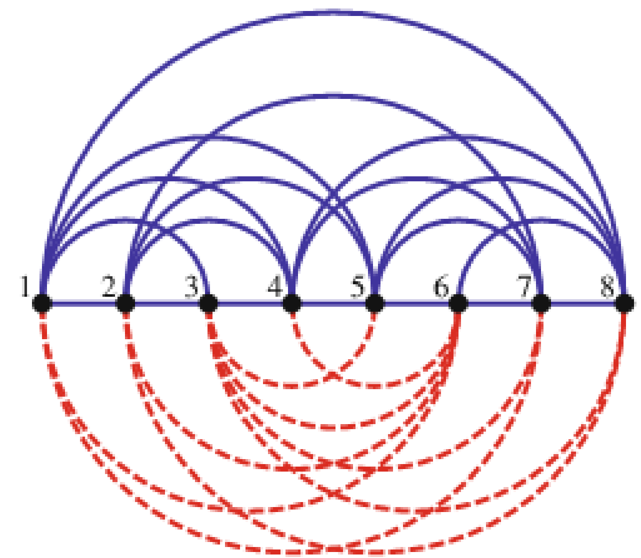

* Idea: average again, and consider $(\leq\leq k)$-edges:

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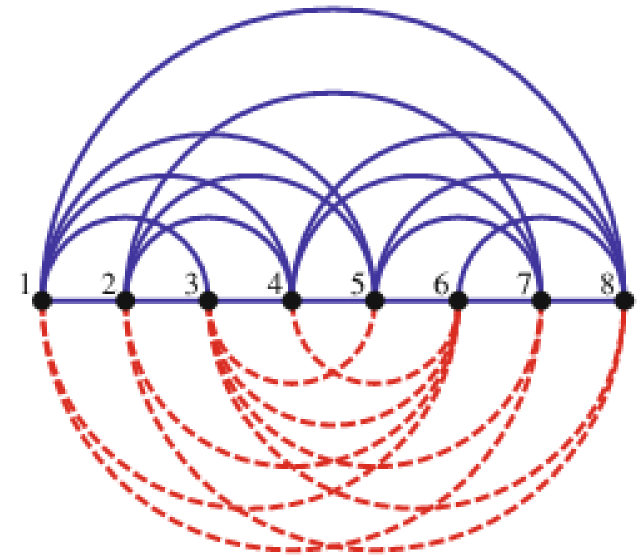

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$$\boxed{\mathrm{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 3} E_{\leq\leq k}(D) + O(n^3)}$$

Optimal lower bounds

∗ 2-page drawings

[Ábrego, Aichholzer, Fernández-Merchant, R., Salazar, 2012]

Optimal lower bounds

- 2-page drawings

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- Monotone drawings

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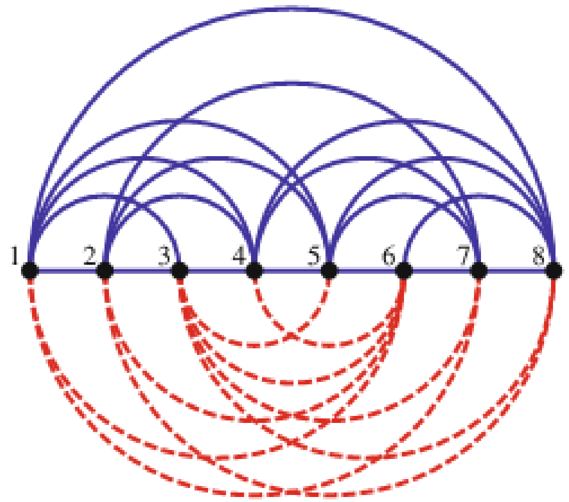
  [Ábrego, Aichholzer, Fernández-Merchant, R., Salazar, 2013]

* In the rest of the talk:

  Sketch of the proof for a slightly more general family: shellable drawings.

Main ideas of the proof

* Convex hull of a drawing: A vertex (or an edge) is in the convex hull of a drawing D if it is visible from infinity.



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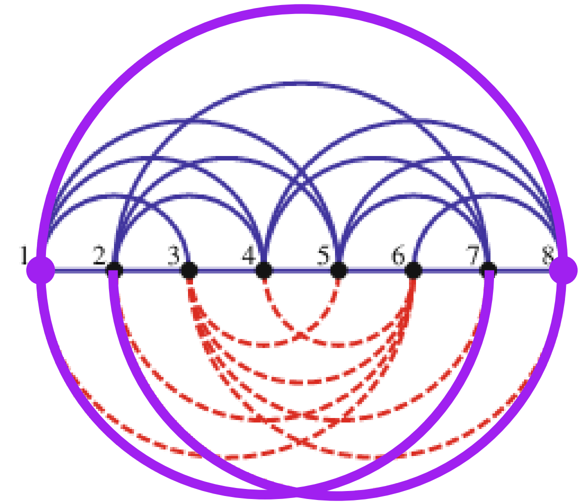

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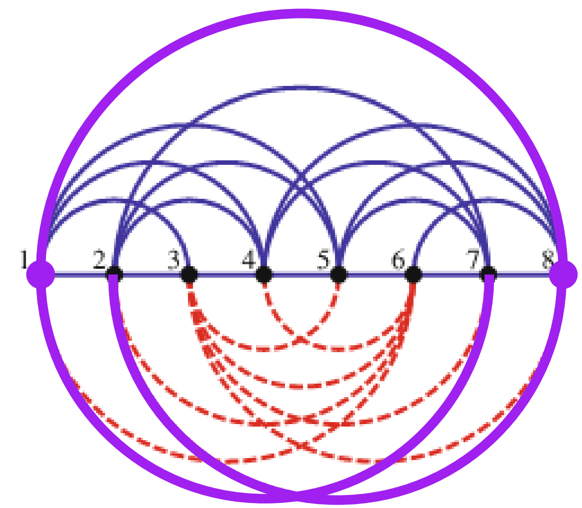

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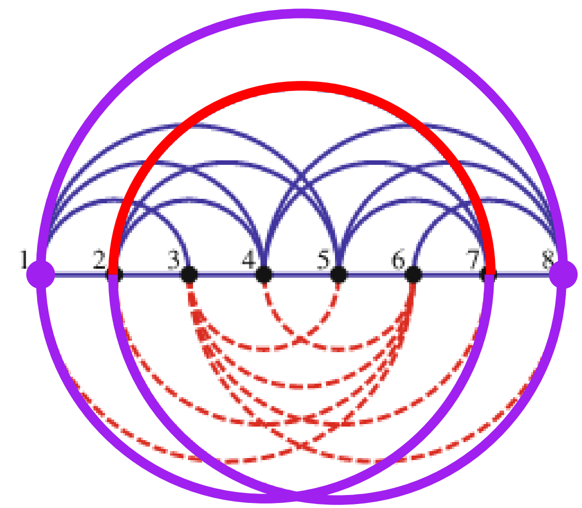

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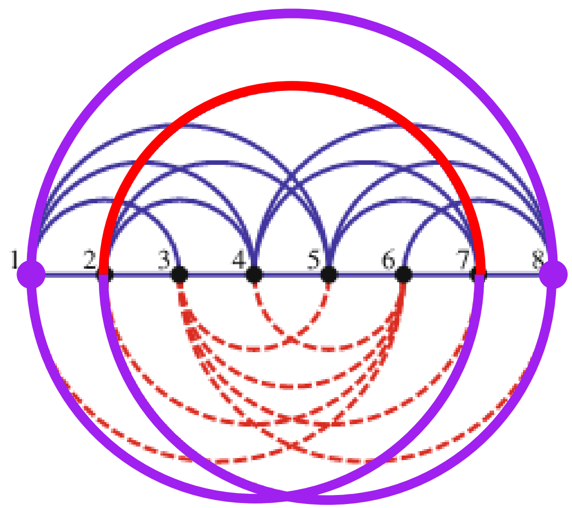
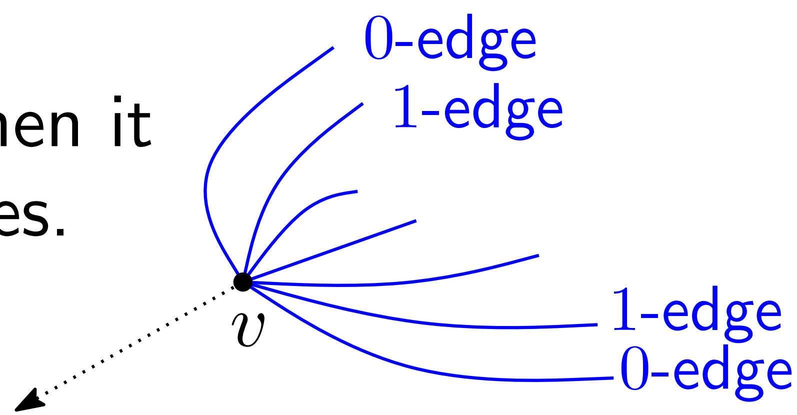


1. if an edge is in the convex hull, then it is a 0-edge (the converse is not true).

2. if a vertex is in the convex hull, then it is adjacent to $2(k+1)$ $(\leq k)$-edges.



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induction
hypothesis

j-edges adjacent
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* A j-edge of D' is an $\leq k$-invariant edge if it is also a j-edge of D (for $j \leq k$).

Finding invariant edges

* Let D_{uv} be the subdrawing of D obtained when vertices $1, 2, \dots, u-1$ and $v+1, v+2, \dots, n$ have been removed.
* We say that D is shellable if there exists a labelling of the vertices such that for all $u < v$, vertices u and v are in the convex hull of D_{uv}

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- We say that D is shellable if there exists a labelling of the vertices such that for all $u < v$, vertices u and v are in the convex hull of D_{uv}

- Theorem: If D is a shellable drawing, then

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$$

(And therefore $\mathrm{cr}(D) \geq Z(n)$.)

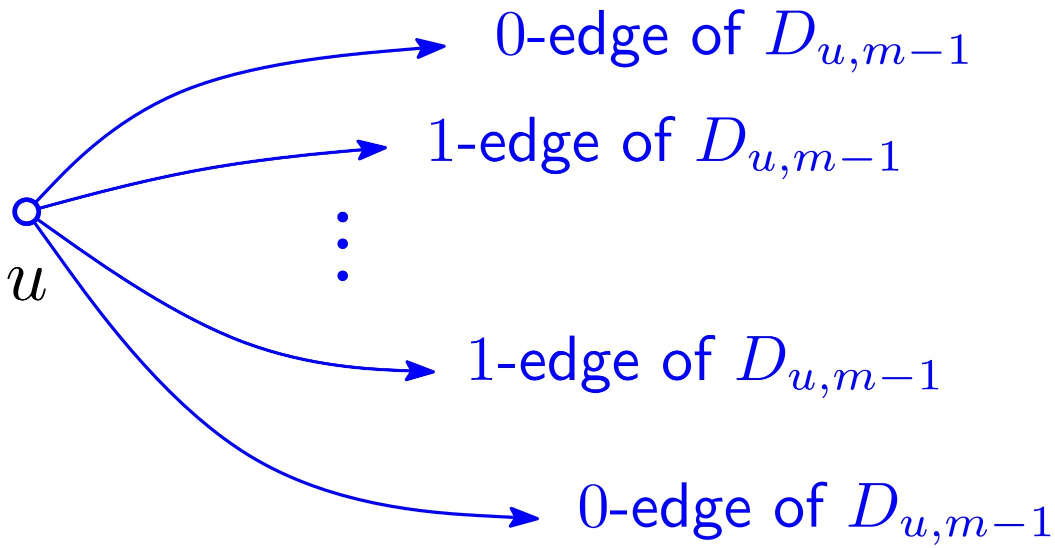
Finding invariant edges

Induction step:

$D = D_{1m}$

$D' = D_{1,m-1}$

Finding invariant edges



Induction step:

$D = D_{1m}$

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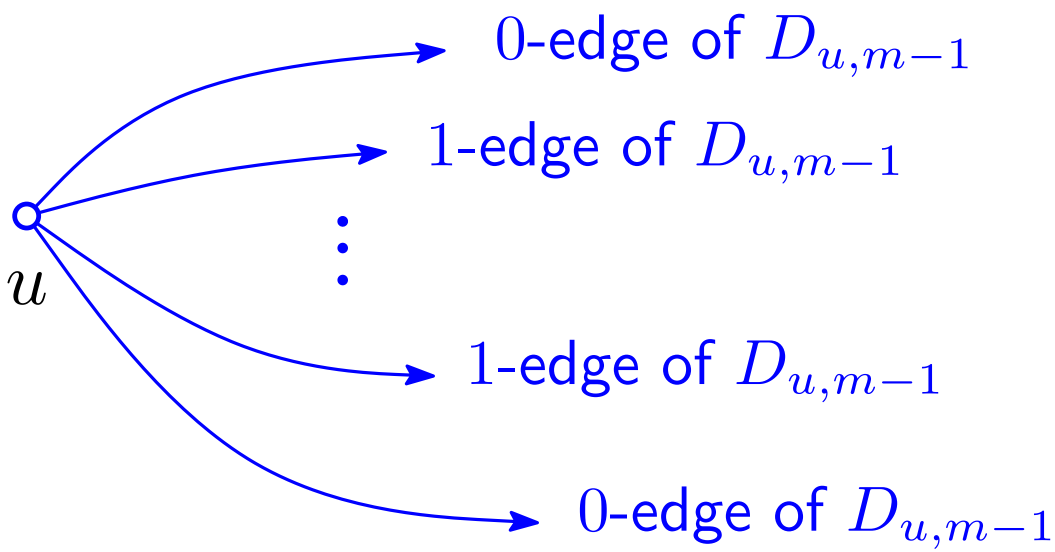
Finding invariant edges



Induction step:

$D = D_{1m}$

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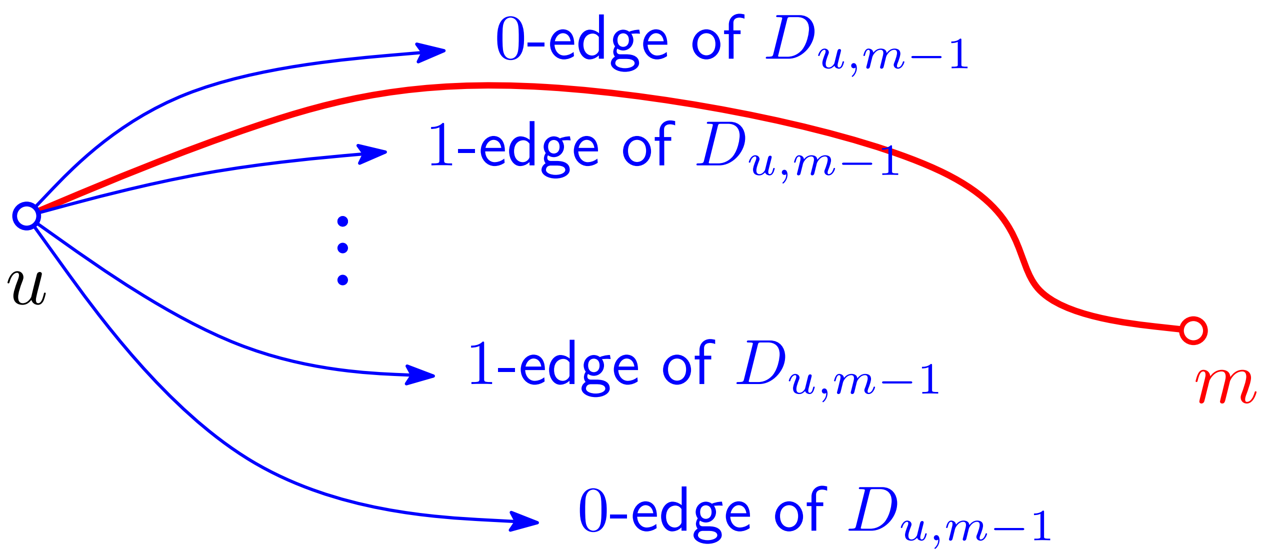
* If e is a 0-edge of $D_{u,m-1}$ adjacent to u, then e is an $\leq (u-1)$-edge of $D_{1,m-1}$.

Finding invariant edges



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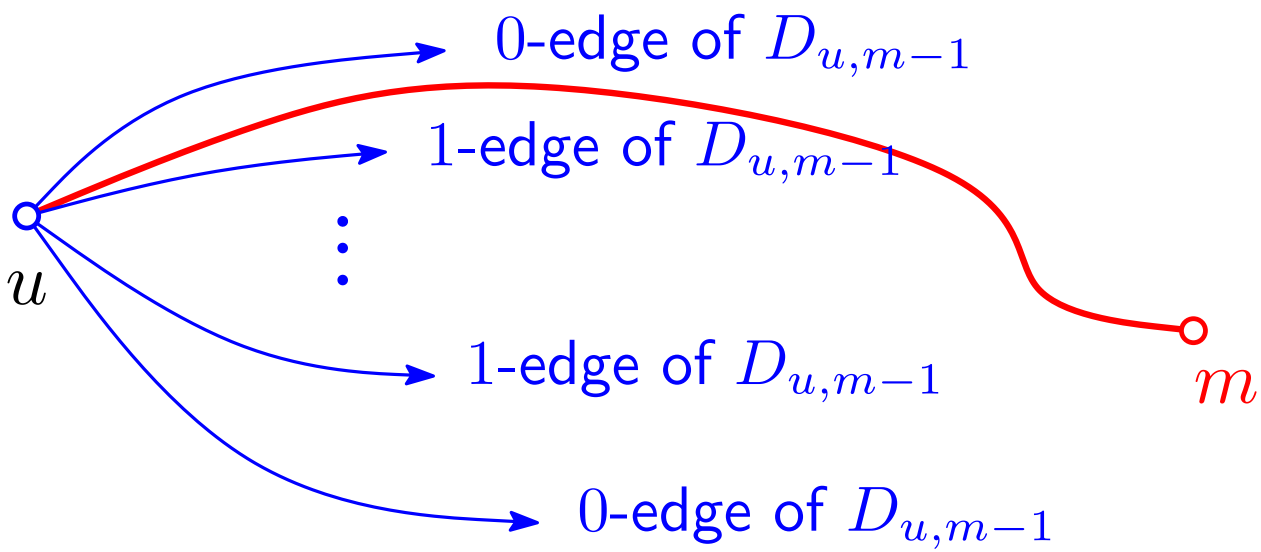
Finding invariant edges



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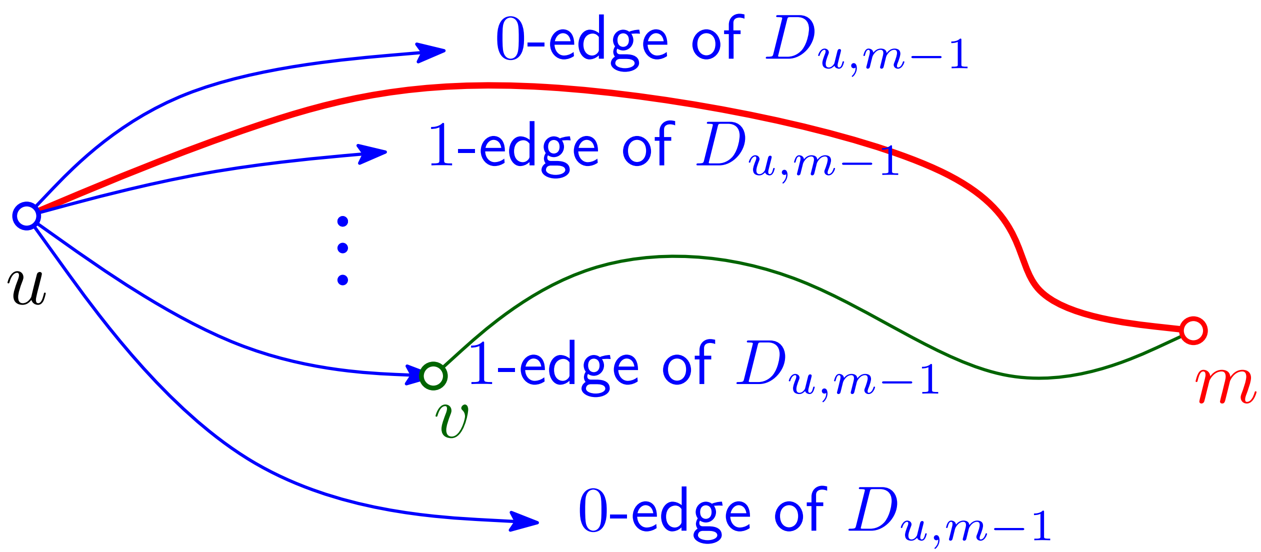
- ∗ If e is a 0-edge of $D_{u,m-1}$ adjacent to u, then e is an $\leq (u-1)$-edge of $D_{1,m-1}$.
- ∗ Consider now edge um
- ∗ Sweep edges starting at u: all the edges that we find before reaching um (or half of the edges) are invariant.

Finding invariant edges



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Finding invariant edges

* Invariant edges starting at u:

$$\left.\begin{array}{l} \text{one } \leq (u-1)\text{-edge} \\ \text{one } \leq u\text{-edge} \\ \quad\vdots \\ \text{one } \leq k\text{-edge} \end{array}\right\} \quad k-u+2 \text{ invariant } \leq k\text{-edges starting at } u.$$

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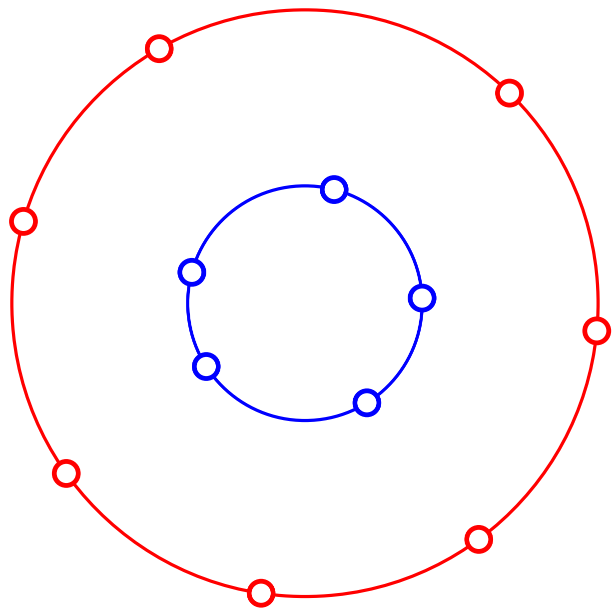
* Considering $u = 1, \ldots, k$, we get $E_{\leq\leq k}(D, D') \geq \binom{k+2}{2}$

* $E_{\leq\leq k}(D) = E_{\leq\leq k-1}(D') + 2\binom{k+2}{2} + E_{\leq k}(D, D')$

$$\Downarrow$$

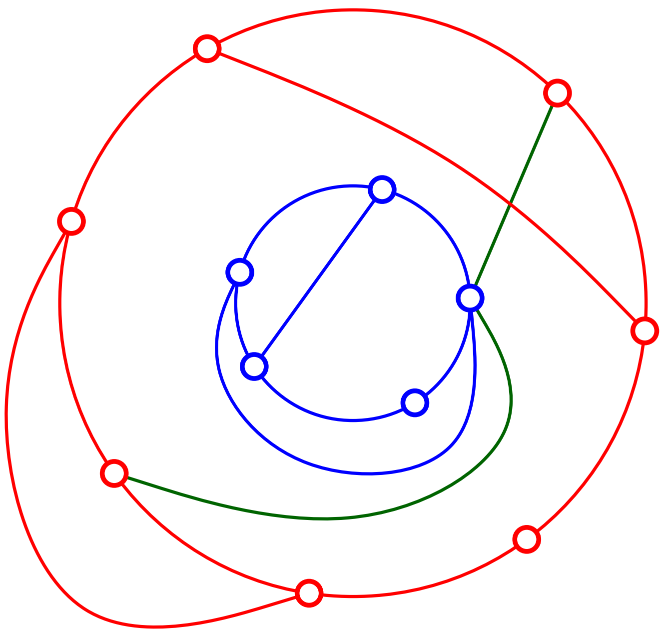
$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3} \longrightarrow \mathrm{cr}(D) \geq Z(n)$$

Cylindrical drawings



A drawing is cylindrical if it contains two crossing-free cycles spanning the set of vertices.

Cylindrical drawings



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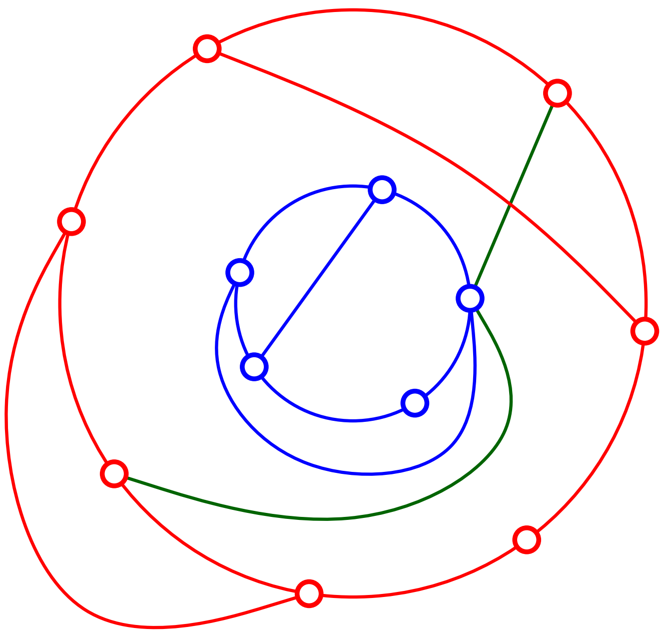
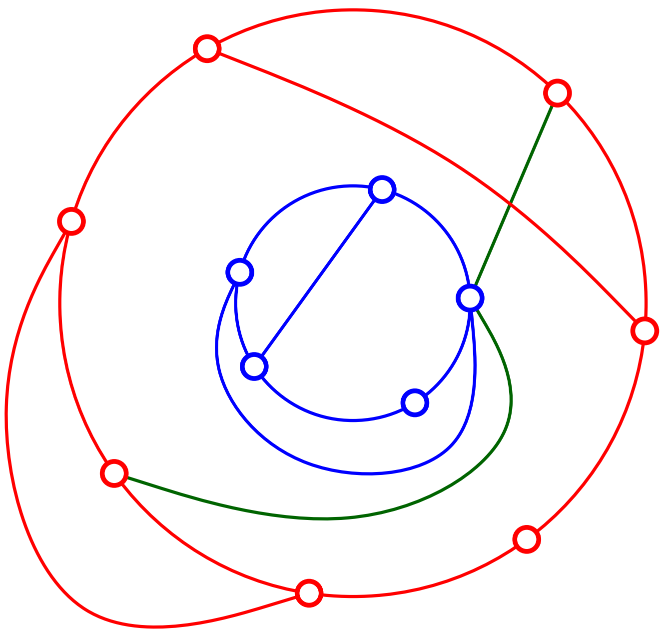
Cylindrical drawings



A drawing is cylindrical if it contains two crossing-free cycles spanning the set of vertices.

Partial results for equal size sets [Richter-Thomassen'97]

Cylindrical drawings



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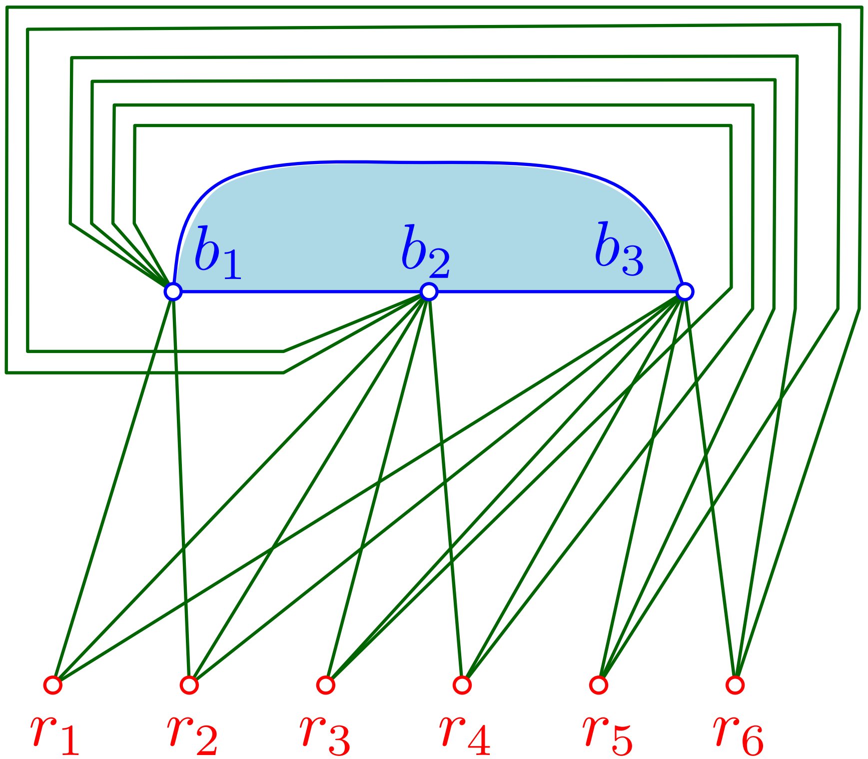
Partial results for equal size sets [Richter-Thomassen'97]

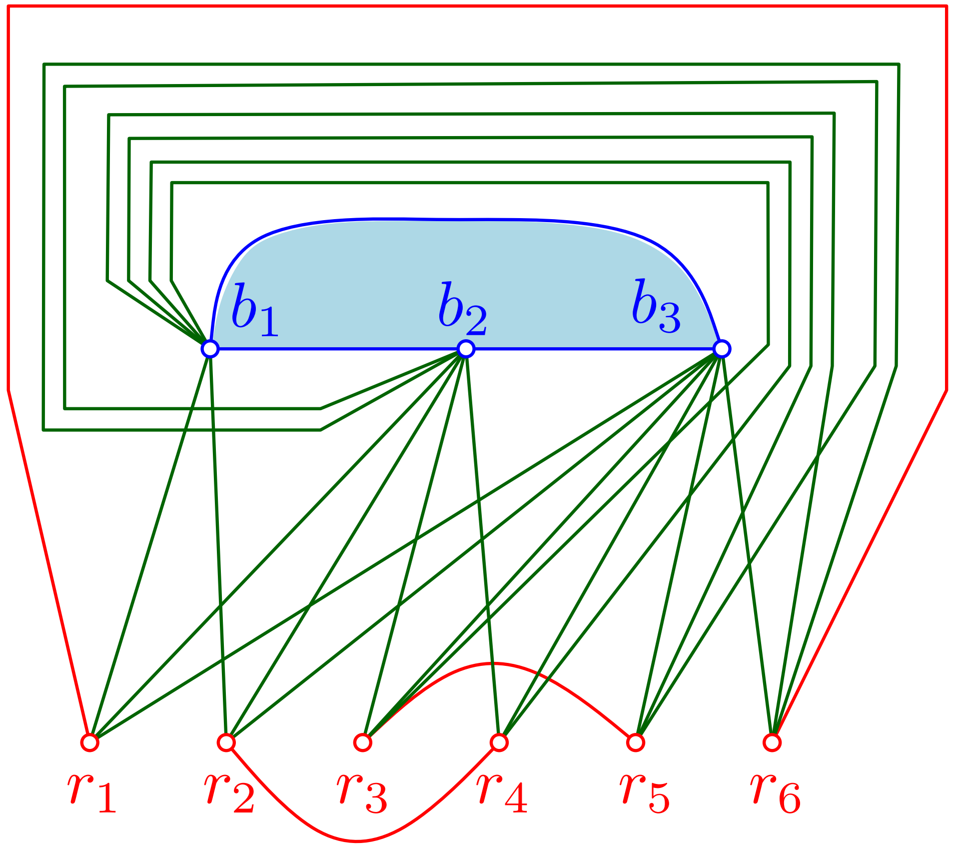
* Optimal cylindrical drawings need not to be shellable, but previous proof can be extended to them.

Optimal cylindrical drawings



$$|B| \leq |R|$$

Optimal cylindrical drawings



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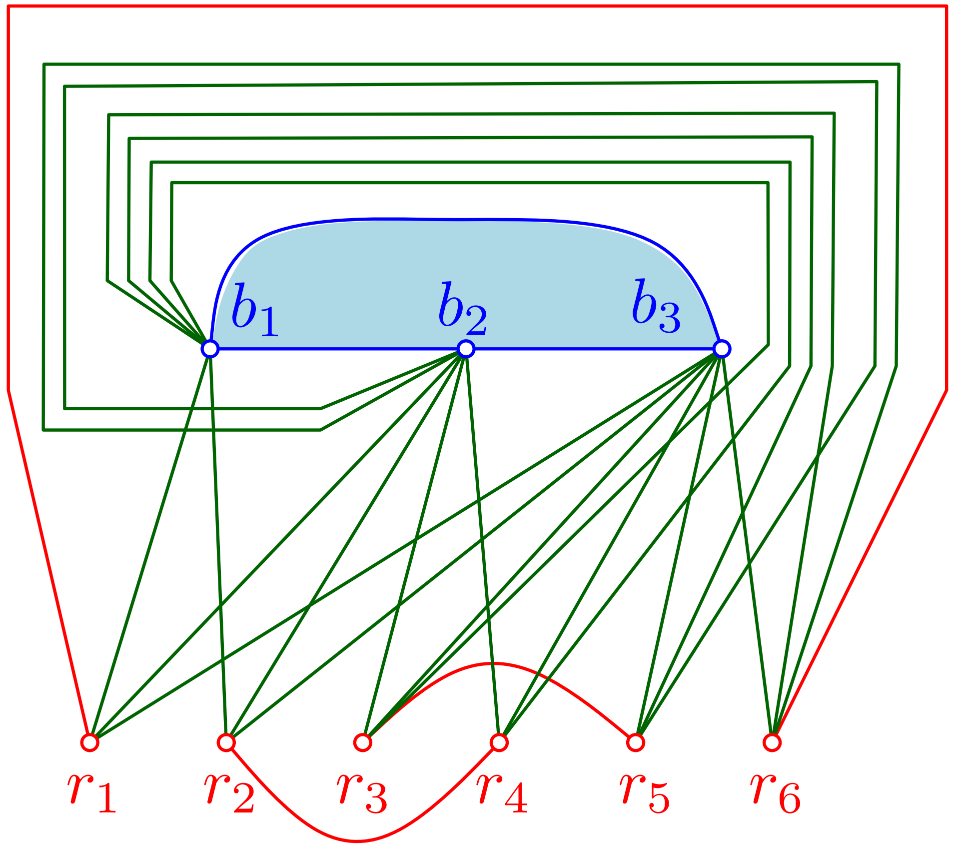

$|B| \leq |R|$

* Idea:
  1. process first red points.
  2. when we reach the blue set, red edges have been removed, and we know the configuration of green and blue ones, so we can find the invariant $\leq k$-edges.

Conclusions

* Two known families of optimal drawings:
  - 2-page drawings
  - cylindrical drawings

  Lower bound known for those families.

Conclusions

* Two known families of optimal drawings:
  - ▷ 2-page drawings
  - ▷ cylindrical drawings

  Lower bound known for those families.

* Open problems:
  - ▷ other families of optimal drawings?
  - ▷ prove that they are really optimal!

Recent developments on the crossing number of K_n

Thank you for your attention

Gracias por vuestra atención

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